German National Olympiad 2014

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–	Day 1
1	For which non-negative integers n is $$K = 5^{2n+3} + 3^{n+3} \cdot 2^n$$ prime?
2	For a positive integer n, let y_n be the number of n-digit positive integers containing only the digits $2, 3, 5, 7$ and which do not have a 5 directly to the right of a 2. If $r \geq 1$ and $m \geq 2$ are integers, prove that y_{m-1} divides y_{rm-1}.
3	Given two positive integers n and k, we say that k is [i]n-ergetic[/i] if: However the elements of $M = \{1, 2, \ldots, k\}$ are coloured in red and green, there exist n not necessarily distinct integers of the same colour whose sum is again an element of M of the same colour. For each positive integer n, determine the least n-ergetic integer, if it exists.
–	Day 2
4	For real numbers x, y and z, solve the system of equations: $$x^3 + y^3 = 3y + 3z + 4$$ $$y^3 + z^3 = 3z + 3x + 4$$ $$x^3 + z^3 = 3x + 3y + 4$$
5	There are 9 visually indistinguishable coins, and one of them is fake and thus lighter. We are given 3 indistinguishable balance scales to find the fake coin; however, one of the scales is defective and shows a random result each time. Show that the fake coin can still be found with 4 weighings.
6	Let $ABCD$ be a circumscribed quadrilateral and M the centre of the incircle. There are points P and Q on the lines MA and MC such that $\angle CBA = 2\angle QBP$. Prove that $\angle ADC = 2\angle PDQ$.