German National Olympiad 2008

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by sqrtX, ZetaX

– Day 1

1 Find all real numbers x such that

$$\sqrt{x+1}+\sqrt{x+3}=\sqrt{2x-1}+\sqrt{2x+1}.$$

2 The triangle $\triangle SFA$ has a right angle at F. The points P and Q lie on the line SF such that the point P lies between S and F and the point F is the midpoint of the segment $[PQ]$. The circle k_1 is th incircle of the triangle $\triangle SPA$. The circle k_2 lies outside the triangle $\triangle SQA$ and touches the segment $[QA]$ and the lines SQ and SA.
Prove that the sum of the radii of the circles k_1 and k_2 equals the length of $[FA]$.

3 Find all functions f defined on non-negative real numbers having the following properties:
(i) For all non-negative x it is $f(x) \geq 0$.
(ii) It is $f\left(1\right)=\frac{1}{2}$.
(iii) For all non-negative numbers x, y it is $f\left(y \cdot f(x)\right) \cdot f(x)=f(x+y)$.

– Day 2

4 Find the smallest constant C such that for all real x, y

$$1+(x+y)^2 \leq C \cdot (1+x^2) \cdot (1+y^2)$$

holds.

5 Inside a square of sidelength 1 there are finitely many disks that are allowed to overlap. The sum of all circumferences is 10. Show that there is a line intersecting or touching at least 4 disks.

6 Find all real numbers x such that $4x^5-7$ and $4x^{13}-7$ are both perfect squares.