Spain Mathematical Olympiad 1971

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by parmenides51

– Day 1

1 Calculate

$$\sum_{k=5}^{k=49} \frac{11_{(k}}{2\sqrt[3]{1331_{(k}}}$$

knowing that the numbers 11 and 1331 are written in base $k \geq 4$.

2 In a certain geometry we operate with two types of elements, points and lines, related to each other by the following axioms:

I. Given two points A and B, there is a unique line (AB) that passes through both.

II. There are at least two points on a line. There are three points not situated on a straight line.

III. When a point B is located between A and C, then B is also between C and A. (A, B, C are three different points on a line.)

IV. Given two points A and C, there exists at least one point B on the line (AC) of the form that C is between A and B.

V. Among three points located on the same straight line, one at most is between the other two.

VI. If A, B, C are three points not lying on the same line and a is a line that does not contain any of the three, when the line passes through a point on segment [AB] , then it goes through one of the $[BC]$, or it goes through one of the [AC] . (We designate by [AB] the set of points that lie between A and B.)

From the previous axioms, prove the following propositions:
Theorem 1. Between points A and C there is at least one point B.
Theorem 2. Among three points located on a line, one is always between the two others.

3 If $0 < p$, $0 < q$ and $p + q < 1$ prove

$$(px + qy)^2 \leq px^2 + qy^2$$

4 Prove that in every triangle with sides a, b, c and opposite angles A, B, C, is fulfilled (measuring the angles in radians)

$$\frac{aA + bB + cC}{a + b + c} \geq \frac{\pi}{3}$$

Hint: Use $a \geq b \geq c \Rightarrow A \geq B \geq C$.

– Day 2

5 Prove that whatever the complex number z is, it is true that

$$(1 + z^{2^n})(1 - z^{2^n}) = 1 - z^{2^{n+1}}.$$

Writing the equalities that result from giving n the values $0, 1, 2, ...$ and multiplying them, show that for $|z| < 1$ holds

$$\frac{1}{1-z} = \lim_{k\to\infty} (1 + z)(1 + z^2)(1 + z^{2^2})...(1 + z^{2^k}).$$

6 The velocities of a submerged and surfaced submarine are, respectively, v and kv. It is situated at a point P at 30 miles from the center O of a circle of 60 mile radius. The surveillance of an enemy squadron forces him to navigate submerged while inside the circle. Discuss, according to the values of k, the fastest path to move to the opposite end of the diameter that passes through P . (Consider the case particular $k = \sqrt{5}$.)

7 Transform by inversion two concentric and coplanar circles into two equal.

8 Among the $2n$ numbers $1, 2, 3, ..., 2n$ are chosen in any way $n + 1$ different numbers. Prove that among the chosen numbers there are at least two, such that one divides the other.