German National Olympiad 2013

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– Day 1

1 Find all positive integers n such that n^2+2^n is square of an integer.

2 Let α be a real number with $\alpha > 1$. Let the sequence (a_n) be defined as

$$a_n = 1 + \sqrt[\alpha]{2 + \sqrt[\alpha]{3 + \ldots + \sqrt[\alpha]{n + \sqrt[\alpha]{n+1}}}}$$

for all positive integers n. Show that there exists a positive real constant C such that $a_n < C$ for all positive integers n.

3 Given two circles k_1 and k_2 which intersect at Q and Q'. Let P be a point on k_2 and inside of k_1 such that the line PQ intersects k_1 in a point $X \neq Q$ and such that the tangent to k_1 at X intersects k_2 in points A and B. Let k be the circle through A, B which is tangent to the line through P parallel to AB.
Prove that the circles k_1 and k are tangent.

– Day 2

4 Let $ABCDEFGH$ be a cube of sidelength a and such that AG is one of the space diagonals. Consider paths on the surface of this cube. Then determine the set of points P on the surface for which the shortest path from P to A and from P to G have the same length l. Also determine all possible values of l depending on a.

5 Five people form several commissions to prepare a competition. Here any commission must be nonempty and any two commissions cannot contain the same members. Moreover, any two commissions have at least one common member.
There are already 14 commissions. Prove that at least one additional commission can be formed.

6 Define a sequence (a_n) by $a_1 = 1, a_2 = 2$, and $a_{k+2} = 2a_{k+1} + a_k$ for all positive integers k. Determine all real numbers $\beta > 0$ which satisfy the following conditions:

(A) There are infinitely pairs of positive integers (p, q) such that $\left|\frac{p}{q} - \sqrt{2}\right| < \frac{\beta}{q^2}$.

(B) There are only finitely many pairs of positive integers (p, q) with $\left|\frac{p}{q} - \sqrt{2}\right| < \frac{\beta}{q^2}$ for which there is no index k with $q = a_k$.