Spain Mathematical Olympiad 1973

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– Day 1

1 Given the sequence (a_n), in which $a_n = \frac{1}{4}n^4 - 10n^2(n-1)$, with $n = 0, 1, 2, ...$ Determine the smallest term of the sequence.

2 Determine all solutions of the system

$$\begin{cases} 2x - 5y + 11z - 6 = 0 \\ -x + 3y - 16z + 8 = 0 \\ 4x - 5y - 83z + 38 = 0 \\ 3x + 11y - z + 9 > 0 \end{cases}$$

in which the first three are equations and the last one is a linear inequality.

3 The sequence (a_n) of complex numbers is considered in the complex plane, in which is:

$$a_0 = 1, \;\; a_n = a_{n-1} + \frac{1}{n}(\cos 45^o + i \sin 45^o)^n.$$

Prove that the sequence of the real parts of the terms of (a_n) is convergent and its limit is a number between 0.85 and 1.15.

4 Let C and C' be two concentric circles of radii r and r' respectively. Determine how much the quotient r'/r must be worth so that in the limited crown (annulus) through C and C' there are eight circles C_i , $i = 1, ..., 8$, which are tangent to C and to C' , and also that C_i is tangent to C_{i+1} for $i = 1, ..., 7$ and C_8 tangent to C_1 .

– Day 2

5 Consider the set of all polynomials of degree less than or equal to 4 with rational coefficients.
a) Prove that it has a vector space structure over the field of numbers rational.
b) Prove that the polynomials $1, x-2, (x-2)^2, (x-2)^3$ and $(x-2)^4$ form a base of this space.
c) Express the polynomial $7 + 2x - 45x^2 + 3x^4$ in the previous base.

6 An equilateral triangle of altitude 1 is considered. For every point P on the interior of the triangle, denote by x, y, z the distances from the point P to the sides of the triangle.
a) Prove that for every point P inside the triangle it is true that $x + y + z = 1$.

b) For which points of the triangle does it hold that the distance to one side is greater than the sum of the distances to the other two?
c) We have a bar of length 1 and we break it into three pieces. find the probability that with these pieces a triangle can be formed.

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7 The two points $P(8,2)$ and $Q(5,11)$ are considered in the plane. A mobile moves from P to Q according to a path that has to fulfill the following conditions: The moving part of P and arrives at a point on the x-axis, along which it travels a segment of length 1, then it departs from this axis and goes towards a point on the y axis, on which travels a segment of length 2, separates from the y axis finally and goes towards the point Q. Among all the possible paths, determine the one with the minimum length, thus like this same length.

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8 In a three-dimensional Euclidean space, by $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, $\overrightarrow{u_3}$ are denoted the three orthogonal unit vectors on the x, y, and z axes, respectively.
a) Prove that the point $P(t) = (1-t)\overrightarrow{u_1} + (2-3t)\overrightarrow{u_2} + (2t-1)\overrightarrow{u_3}$, where t takes all real values, describes a straight line (which we will denote by L).
b) What describes the point $Q(t) = (1-t^2)\overrightarrow{u_1} + (2-3t^2)\overrightarrow{u_2} + (2t^2-1)\overrightarrow{u_3}$ if t takes all the real values?
c) Find a vector parallel to L.
d) For what values of t is the point $P(t)$ on the plane $2x+3y+2z+1=0$?
e) Find the Cartesian equation of the plane parallel to the previous one and containing the point $Q(3)$.
f) Find the Cartesian equation of the plane perpendicular to L that contains the point $Q(2)$.
