Spain Mathematical Olympiad 1974

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– Day 1

1 It is known that a regular dodecahedron is a regular polyhedron with 12 faces of equal pentagons and concurring 3 edges in each vertex. It is requested to calculate, reasonably,
a) the number of vertices,
b) the number of edges,
c) the number of diagonals of all faces,
d) the number of line segments determined for every two vertices,
d) the number of diagonals of the dodecahedron.

2 In a metallic disk, a circular sector is removed, so that with the remaining can form a conical glass of maximum volume. Calculate, in radians, the angle of the sector that is removed.

En un disco metalico se quita un sector circular, de modo que con la parte restante se pueda formar un vaso c ´onico de volumen maximo. Calcular, en radianes, el angulo del sector que se quita.

3 We will designate by $Z_{(5)}$ a certain subset of the set Q of the rational numbers . A rational belongs to $Z_{(5)}$ if and only if there exist equal fraction to this rational such that 5 is not a divisor of its denominator. (For example, the rational number $13/10$ does not belong to $Z_{(5)}$, since the denominator of all fractions equal to $13/10$ is a multiple of 5. On the other hand, the rational $75/10$ belongs to $Z_{(5)}$ since that $75/10 = 15/12$).
Reasonably answer the following questions:
a) What algebraic structure (semigroup, group, etc.) does $Z_{(5)}$ have with respect to the sum?
b) And regarding the product?
c) Is $Z_{(5)}$ a subring of Q?
d) Is $Z_{(5)}$ a vector space?

4 All three sides of an equilateral triangle are assumed to be reflective (except in the vertices), in such a way that they reflect the rays of light located in their plane, that fall on them and that come out of an interior point of the triangle.
Determine the path of a ray of light that, starting from a vertex of the triangle reach another vertex of the same after reflecting successively on the three sides. Calculate the length of the path followed by the light assuming that the side of the triangle measures 1 m.

– Day 2

5 Let $(G, \cdot)$ be a group and e an identity element. Prove that if all elements x of G satisfy $x \cdot x = e$ then $(G, \cdot)$ is abelian (that is, commutative).

6 Two chords are drawn in a circle of radius equal to unit, AB and AC of equal length.
a) Describe how you can construct a third chord DE that is divided into three equal parts by the intersections with AB and AC.
b) If $AB = AC = \sqrt{2}$, what are the lengths of the two segments that the chord DE determines in AB?

7 A tank has the shape of a regular hexagonal prism, whose bases are 1 m on a side and its height is 10 m. The lateral edges are placed in an oblique position and is partially filled with 9 m^3 of water. The plane of the free surface of the water cuts to all lateral edges. One of them is left with a part of 2 m under water. What part is under water on the opposite side edge of the prism?

8 The sides of a convex regular polygon of $L+M+N$ sides are to be given draw in three colors: L of them with a red stroke, M with a yellow stroke, and N with a blue. Express, through inequalities, the necessary and sufficient conditions so that there is a solution (several, in general) to the problem of doing it without leaving two adjacent sides drawn with the same color.