German National Olympiad 2012

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–	Day 1
1	Define a sequence (a_n) by $a_0=-4, a_1=-7$ and $a_{n+2}=5a_{n+1}-6a_n$ for $n\geq 0$. Prove that there are infinitely many positive integers n such that a_n is composite.
2	Find the maximal number of edges a connected graph G with n vertices may have, so that after deleting an arbitrary cycle, G is not connected anymore.
3	Let ABC a triangle and k a circle such that: (1) The circle k passes through A and B and touches the line AC. (2) The tangent to k at B intersects the line AC in a point $X\neq C$. (3) The circumcircle ω of BXC intersects k in a point $Q\neq B$. (4) The tangent to ω at X intersects the line AB in a point Y. Prove that the line XY is tangent to the circumcircle of BQY.
–	Day 2
4	Let a,b be positive real numbers and $n\geq 2$ a positive integer. Prove that if $x^n\leq ax+b$ holds for a positive real number x, then it also satisfies the inequality $x<\sqrt[n-1]{2a}+\sqrt[n]{2b}$.
5	Let a,b be the lengths of two nonadjacent edges of a tetrahedron with inradius r. Prove that $$r<\frac{ab}{2(a+b)}.$$
6	Let a_1 and a_2 be postive real numbers. Let $a_{n+2}=1+\frac{a_{n+1}}{a_n}$ Prove that $\lvert a_{2012}-2\rvert<10^{-200}$