all 3 levels

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–	Level 3
–	Day 1
1	Prove that there exists infinitely many positive integers n for which the equation $$x^2+y^{11}-z^{2022!}=n$$ has no solution (x,y,z) over the integers.
2	Let ABC be an acute triangle with $AB<AC$. Let D,E,F be the feet of the altitudes relatives to the vertices A,B,C, respectively. The circumcircle Γ of AEF cuts the circumcircle of ABC at A and M. Assume that BM is tangent to Γ. Prove that M, F and D are collinear.
3	Let n be a positive integer. Given a sequence of nonnegative real numbers $x_1,\dots,x_n$ we define the *transformed sequence* $y_1,\dots,y_n$ as follows: the number y_i is the greatest possible value of the average of consecutive terms of the sequence that contain x_i. For example, the transformed sequence of $2,4,1,4,1$ is $3,4,3,4,5/2$. Prove that a) For every positive real number t, the number of y_i such that $y_i>t$ is less than or equal to $\frac{2}{t}(x_1+\dots+x_n)$. b) The inequality $\frac{y_1+\dots+y_n}{32n}\le\sqrt{\frac{x_1^2+\dots+x_n^2}{32n}}$ holds.
–	Day 2
4	Let ABC be a triangle with incenter I. Let D be the point of intersection between the incircle and the side BC, the points P and Q are in the rays IB and IC, respectively, such that $\angle IAP=\angle CAD$ and $\angle IAQ=\angle BAD$. Prove that $AP=AQ$.
5	Let $n\ge 4$ and k be positive integers. We consider n lines in the plane between which there are not two parallel nor three concurrent. In each of the $\frac{n(n-1)}{2}$ points of intersection of these lines, k coins are placed. Ana and Beto play the following game in turns: each player, in turn, chooses one of those points that does not share one of the n lines with the point chosen immediately before by the other player, and removes a coin from that point. Ana starts and can choose any point. The player who cannot make his move loses. Determine based on n and k who has a winning strategy.

6 In a board, the positive integer N is written. In each round, Olive can realize any one of the following operations:
I - Switch the current number by a positive multiple of the current number.
II - Switch the current number by a number with the same digits of the current number, but the digits are written in another order(leading zeros are allowed). For instance, if the current number is 2022, Olive can write any of the following numbers $222, 2202, 2220.$
Determine all the positive integers N, such that, Olive can write the number 1 after a finite quantity of rounds.

– Level 2

– Day 1

1 In how many ways can the numbers from 2 to 2022 be arranged so that the first number is a multiple of 1, the second number is a multiple of 2, the third number is a multiple of 3, and so on untile the last number is a multiple of 2021?

2 Let $m, n \geq 2$. One needs to cover the table $m \times n$ using only the following tiles:
Tile 1 - A square 2×2.
Tile 2 - A L-shaped tile with five cells, in other words, the square 3×3 **without** the upper right square 2×2.
Each tile 1 covers exactly 4 cells and each tile 2 covers exactly 5 cells. Rotation is allowed.
Determine all pairs (m, n), such that the covering is possible.

3 Let ABC be a triangle with $AB < AC$. There are two points X and Y on the angle bisector of $B\widehat{A}C$ such that X is between A and Y and BX is parallel to CY. Let Z be the reflection of X with respect to BC. Line YZ cuts line BC at point P. If line BY cuts line CX at point K, prove that $KA = KP$.

– Day 2

4 Let $ABCD$ be a parallelogram and M is the intersection of AC and BD. The point N is inside of the $\triangle AMB$ such that $\angle AND = \angle BNC$. Prove that $\angle MNC = \angle NDA$ and $\angle MND = \angle NCB$.

5 Let n be a positive integer. The numbers $1, 2, 3, \dots, 4n$ are written in a board. Olive wants to make some "couples" of numbers, such that the product of the numbers in each couple is a perfect square. Each number is in, at most, one couple and the two numbers in each couple are distincts.
Determine, for each positive integer n, the maximum number of couples that Olive can write.

6 Let $N(a, b)$ be the number of ways to cover a table $a \times b$ with domino tiles. Let $M(a, 2b+1)$ be the number of ways to cover a table $a \times 2b+1$ with domino tiles, such that there are **no** vertical tile in the central column. Prove that

$$M(2m, 2n+1) = 2^m \cdot N(2m, n) \cdot N(2m, n-1)$$

– Level 1

– Day 1

1 Find three consecutive odd numbers a, b, c such that $a^2 + b^2 + c^2$ is a four digit number with four equal digits.

2 Four teams A, B, C and D play a football tournament in which each team plays exactly two times against each of the remaining three teams (there are 12 matches). In each matchif it's a tie each team gets 1 point and if it isn't a tie then the winner gets 3 points and the loser gets 0 points.
At the end of the tournament the teams A, B and C have 8 points each. Determine all possible points of team D.

3 On the table there are N cards. Each card has an integer number written on it.
Beto performs the following operation several times: he chooses two cards from the table, calculates the difference between the numbers written on them, writes the result on his notebook and removes those two cards from the table. He can perform this operation as many times as he wants, as long as there are at least two cards on the table.
After this, Beto multiplies all the numbers that he wrote on his notebook. Beto's goal is that the result of this multiplication is a multiple of 7^{100}.
Find the minimum value of N such that Beto can always achieve his goal, no matter what the numbers on the cards are.

– Day 2

4 Let L be the number formed by 2022 digits equal to 1, that is, $L = 1111\ldots111$.
Compute the sum of the digits of the number $9L^2 + 2L$.

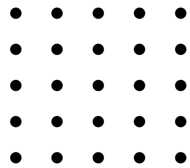
5 Let $ABCDEFGHI$ be a regular polygon with 9 sides and the vertices are written in the counterclockwise and let $ABJKLM$ be a regular polygon with 6 sides and the vertices are written in the clockwise. Prove that $\angle HMG = \angle KEL$.
Note: The polygon $ABJKLM$ is inside of $ABCDEFGHI$.

6 In Vila Par, all the truth coins weigh an even quantity of grams and the false coins weigh an odd quantity of grams. The eletronic device only gives the parity of the weight of a set of coins. If there are 2020 truth coins and 2 false coins, detemine the least k, such that, there exists a strategy that allows to identify the two false coins using the eletronic device, at most, k times.

– Level A

– Day 1

1 In the blackboard there are drawn 25 points as shown in the figure.
Gastón must choose 4 points that are vertices of a square.
In how many different ways can he make this choice?



2 Eight teams play a rugby tournament in which each team plays exactly one match against each of the remaining seven teams. In each match, if it's a tie each team gets 1 point and if it isn't a tie then the winner gets 2 points and the loser gets 0 points. After the tournament it was observed that each of the eight teams had a different number of points and that the number of points of the winner of the tournament was equal to the sum of the number of points of the last four teams.
Give an example of a tournament that satisfies this conditions, indicating the number of points obtained by each team and the result of each match.

3 On the table there are several cards. Each card has an integer number written on it.
Beto performs the following operation several times: he chooses two cards from the table, calculates the difference between the numbers written on them, writes the result on his notebook and removes those two cards from the table. He can perform this operation as many times as he wants, as long as there are at least two cards on the table.
After this, Beto multiplies all the numbers that he wrote on his notebook. Beto's goal is that the result of this multiplication is a multiple of 7^{100}.

a) Prove that if there are 207 cards initially on the table then Beto can always achieve his goal, no matter what the numbers on the cards are.
b) If there are 128 cards initially on the table, is it true that Beto can always achieve his goal?

– Day 2

5 The quadrilateral $ABCD$ has the following equality $\angle ABC = \angle BCD = 150^\circ$. Moreover, $AB = 18$ and $BC = 24$, the equilateral triangles $\triangle APB, \triangle BQC, \triangle CRD$ are drawn outside the quadrilateral. If $P(X)$ is the perimeter of the polygon X, then the following equality is true $P(APQRD) = P(ABCD) + 32$. Determine the length of the side CD.

6 A sequence of numbers is *platense* if the first number is greater than 1, and $a_{n+1} = \frac{a_n}{p_n}$ which p_n is the least prime divisor of a_n, and the sequence ends if $a_n = 1$. For instance, the sequences $864, 432, 216, 108, 54, 27, 9, 3, 1$ and $2022, 1011, 337, 1$ are both sequence platense. A sequence platense is *cuboso* if some term is a perfect cube greater than 1. For instance, the sequence 864 is cuboso, because $27 = 3^3$, and the sequence 2022 is not cuboso, because there is no perfect cube. Determine the number of sequences cuboso which the initial term is less than 2022.