German National Olympiad 2011

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–	Day 1
1	Prove for each non-negative integer n and real number x the inequality $$\sin x \cdot (n \sin x - \sin nx) \geq 0$$
2	The price for sending a packet (a rectangular cuboid) is directly proportional to the sum of its length, width, and height. Is it possible to reduce the cost of sending a packet by putting it into a cheaper packet?
3	Let ABC be an acute triangle and D the foot of the altitude from A onto BC. A semicircle with diameter BC intersects segments AB, AC and AD in the points F, E resp. X. The circumcircles of the triangles DEX and DXF intersect BC in L resp. N other than D. Prove $BN = LC$.
–	Day 2
4	There are two points A and B in the plane. a) Determine the set M of all points C in the plane for which $\|AC\|^2 + \|BC\|^2 = 2 \cdot \|AB\|^2$. b) Decide whether there is a point $C \in M$ such that $\angle ACB$ is maximal and if so, determine this angle.
5	Prove or disprove: $\exists n \in N$, s.t. $324 + 455^n$ is prime.
6	Let $p > 2$ be a prime. Define a sequence $(Q_n(x))$ of polynomials such that $Q_0(x) = 1, Q_1(x) = x$ and $Q_{n+1}(x) = xQ_n(x) + nQ_{n-1}(x)$ for $n \geq 1$. Prove that $Q_p(x) - x^p$ is divisible by p for all integers x.