

German National Olympiad 2010

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– Day 1

1 Given two circles k and l which intersect at two points. One of their common tangents touches k at point K , while the other common tangent touches l at L . Let A and B be the intersections of the line KL with the circles k and l , respectively. Prove that $\overline{AK} = \overline{BL}$.

2 Let a, b, c be pairwise distinct real numbers. Show that

$$\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2 \geq 5.$$

3 An infinite fairytale is a book with pages numbered $1, 2, 3, \dots$ where all natural numbers appear. An author wants to write an infinite fairytale such that a new dwarf is introduced on each page. Afterward, the page contains several discussions between groups of at least two of the already introduced dwarfs. The publisher wants to make the book more exciting and thus requests the following condition:
Every infinite set of dwarfs contains a group of at least two dwarfs, who formed a discussion group at some point as well as a group of the same size for which this is not true.
Can the author fulfill this condition?

– Day 2

4 Find all positive integer solutions for the equation $(3x+1)(3y+1)(3z+1) = 34xyz$
Thx

5 The polynomial $x^8 + x^7$ is written on a blackboard. In a move, Peter can erase the polynomial $P(x)$ and write down $(x+1)P(x)$ or its derivative $P'(x)$. After a while, the linear polynomial $ax+b$ with $a \neq 0$ is written on the board. Prove that $a-b$ is divisible by 49.

6 Let A, B, C, D, E, F, G and H be eight pairwise distinct points on the surface of a sphere. The quadruples $(A, B, C, D), (A, B, F, E), (B, C, G, F), (C, D, H, G)$ and (D, A, E, H) of points are coplanar.
Prove that the quadruple (E, F, G, H) is coplanar as well.
