2020 Israel National Olympiad

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1 Seven identical-looking coins are given, of which four are real and three are counterfeit. The three counterfeit coins have equal weight, and the four real coins have equal weight. It is known that a counterfeit coin is lighter than a real one. In one weighing, one can select two sets of coins and check which set has a smaller total weight, or if they are of equal weight. How many weightings are needed to identify one counterfeit coin?

2 202 participants arrived at a mathematical conference from three countries: Israel, Greece, and Japan.
On the first day of the conference, every pair of participants from the same country shook hands.
On the second day, every pair of participants exactly one of whom was Israeli shook hands.
On the third day, every pair of participants one of whom was Israeli and the other Greek shook hands.
In total 20200 handshakes occurred. How many Israelis participated in the conference?

3 In a convex hexagon $ABCDEF$ the triangles BDF, ACE are equilateral and congruent. Prove that the three lines connecting the midpoints of opposite sides are concurrent.

4 At the start of the day, the four numbers (a_0, b_0, c_0, d_0) were written on the board. Every minute, Danny replaces the four numbers written on the board with new ones according to the following rule: if the numbers written on the board are (a, b, c, d), then Danny first calculates the numbers

$$\begin{aligned} a' &= a + 4b + 16c + 64d \\ b' &= b + 4c + 16d + 64a \\ c' &= c + 4d + 16a + 64b \\ d' &= d + 4a + 16b + 64c \end{aligned}$$

and replaces the numbers (a, b, c, d) with the numbers $(a'd', c'd', c'b', b'a')$.
For which initial quadruples (a_0, b_0, c_0, d_0), will Danny write at some point a quadruple of numbers all of which are divisible by 5780^{5780}?

5 Two triangles ACE, BDF are given which intersect at six points: G, H, I, J, K, L as in the picture. It is known that in each of the quadrilaterals

$$ABIK, BCJL, CDKG, DELH, EFGI$$

it is possible to inscribe a circle. Is it possible for the quadrilateral $FAHJ$ is also circumscribed around a circle?

6 On a circle the numbers from 1 to 6 are written in order, as depicted in the picture. In each move, Lior picks a number a on the circle whose neighbors are b and c and replaces it by the number $\frac{bc}{a}$. Can Lior reach a state in which the product of the numbers on the circle is greater than 10^{100} in

a) at most 100 moves

b) at most 110 moves

7 Let P be a point inside a triangle ABC, d_a, d_b and d_c be distances from P to the lines BC, AC and AB respectively, R be a radius of the circumcircle and r be a radius of the inscribed circle for ΔABC. Prove that:

$$\sqrt{d_a}+\sqrt{d_b}+\sqrt{d_c} \le \sqrt{2R+5r}.$$