An Olympiad for grades 7-12.

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P1 Sophie wrote on a piece of paper every integer number from 1 to 1000 in decimal notation (including both endpoints).
a) Which digit did Sophie write the most?
b) Which digit did Sophie write the least?

P2 Does there exist an infinite sequence of primes $p_1, p_2, p_3, \dots$ for which,

$$p_{n+1} = 2p_n + 1$$

for each n?

P3 Let ABC be a triangle. Let X be the tangency point of the incircle with BC. Let Y be the second intersection point of segment AX with the incircle.
Prove that

$$AX + AY + BC > AB + AC$$

P4 Danny likes seven-digit numbers with the following property: the 1's digit is divisible by the 10's digit, the 10's digit is divisible by the 100's digit, and so on.
For example, Danny likes the number 1133366 but doesn't like 9999993.

Is the amount of numbers Danny likes divisible by 7?

P5 Solve the following equation in positive numbers.

$$(2a+1)(2a^2+2a+1)(2a^4+4a^3+6a^2+4a+1) = 828567056280801$$

P6 21 players participated in a tennis tournament, in which each pair of players played exactly once and each game had a winner (no ties are allowed).
The organizers of the tournament found out that each player won at least 9 games, and lost at least 9. In addition, they discovered cases of three players A, B, C in which A won against B, B won against C and C won against A, and called such triples "problematic".
a) What is the maximum possible number of problematic triples?
b) What is the minimum possible number of problematic triples?

P7 Triangle ABC is given.
The circle ω with center I is tangent at points D, E, F to segments BC, AC, AB respectively.
When ABC is rotated 180 degrees about point I, triangle $A'B'C'$ results.
Lines $AD, B'C'$ meet at U, lines $BE, A'C'$ meet at V, and lines $CF, A'B'$ meet at W.
Line BC meets $A'C', A'B'$ at points D_1, D_2 respectively.
Line AC meets $A'B', B'C'$ at E_1, E_2 respectively.
Line AB meets $B'C', A'C'$ at F_1, F_2 respectively.
Six (not necessarily convex) quadrilaterals were colored orange:

$$AUIF_2, C'FIF_2, BVID_1, A'DID_2, CWIE_1, B'EIE_2$$

Six other quadrilaterals were colored green:

$$AUIE_2, C'FIF_1, BVIF_2, A'DID_1, CWID_2, B'EIE_1$$

Prove that the sum of the green areas equals the sum of the orange areas.