An Olympiad for grades 7-12.

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P1 In a room are several people, some of which always lie and all others always tell the truth. Their ages are pairwise distinct. Each person says one of the following phrases:
"In this room, there is an equal number of truth-sayers older than me and of liars younger than me"
or
"In this room, there is an equal number of truth-sayers younger than me and of liars older than me"
What is the maximum possible number of truth-sayers in the room?
Find an example in which this maximum is achieved and prove a higher number is impossible.

P2 Real nonzero numbers a, b, c, d, e, f, k, m satisfy the equations

$$\frac{a}{b}+\frac{c}{d}+\frac{e}{f}=k$$

$$\frac{b}{c}+\frac{d}{e}+\frac{f}{a}=m$$

$$ad=be=cf$$

Express $\frac{a}{c}+\frac{c}{e}+\frac{e}{a}+\frac{b}{d}+\frac{d}{f}+\frac{f}{b}$ using m and k.

P3 Let w be a circle of diameter 5. Four lines were drawn dividing w into 5 "strips", each of width 1. The strips were colored orange and purple alternatingly, as depicted. Which area is larger: the orange or the purple?

P4 Find all triples (a, b, c) of integers for which the equation

$$x^3-a^2x^2+b^2x-ab+3c=0$$

has three distinct integer roots x_1, x_2, x_3 which are pairwise coprime.

P5 A paper convex quadrilateral will be called **folding** if there are points P, Q, R, S on the interiors of segments AB, BC, CD, DA respectively so that if we fold in the triangles SAP, PBQ, QCR, RDS, they will exactly cover the quadrilateral $PQRS$. In other words, if the folded triangles will cover the quadrilateral $PQRS$ but won't cover each other.
Prove that if quadrilateral $ABCD$ is folding, then $AC \perp BD$ or $ABCD$ is a trapezoid.

P6 Let x, y, z be non-negative real numbers. Prove that:

$$\sqrt{(2x+y)(2x+z)} + \sqrt{(2y+x)(2y+z)} + \sqrt{(2z+x)(2z+y)} \geq$$
$$\geq \sqrt{(x+2y)(x+2z)} + \sqrt{(y+2x)(y+2z)} + \sqrt{(z+2x)(z+2y)}.$$

P7 Gandalf (the wizard) and Bilbo (the assistant) are presenting a magic trick to Nitzan (the audience). While Gandalf leaves the room, Nitzan chooses a number $1 \leq x \leq 2^{2022}$ and shows it to Bilbo. Now bilbo writes on the board a long row of N digits, each of which is 0 or 1. After this Nitzan can, if he wishes, switch the order of two consecutive digits in the row, but only once. Then Gandalf returns to the room, looks at the row, and guesses the number x.

Can Bilbo and Gandalf come up with a strategy that allows Gandalf to guess x correctly no matter how Nitzan acts, if
a) $N = 2500$?
b) $N = 2030$?
c) $N = 2040$?
