An Olympiad for grades 7-12.

www.artofproblemsolving.com/community/c3229056

by Phorphyrion

Date December 14, 2022

P1 2000 people are sitting around a round table. Each one of them is either a truth-sayer (who always tells the truth) or a liar (who always lies). Each person said: ”At least two of the three people next to me to the right are liars”. How many truth-sayers are there in the circle?

P2 The non-negative integers x, y satisfy $\sqrt{x}+\sqrt{x+60}=\sqrt{y}$. Find the largest possible value for x.

P3 A triangle ABC is given together with an arbitrary circle ω. Let α be the reflection of ω with respect to A, β the reflection of ω with respect to B, and γ the reflection of ω with respect to C. It is known that the circles α, β, γ don't intersect each other.
Let P be the meeting point of the two internal common tangents to β, γ (see picture). Similarly, Q is the meeting point of the internal common tangents of α, γ, and R is the meeting point of the internal common tangents of α, β.
Prove that the triangles PQR, ABC are congruent.

P4 For each positive integer n, find all triples a, b, c of real numbers for which

$$\begin{cases} a=b^n+c^n \\ b=c^n+a^n \\ c=a^n+b^n \end{cases}$$

P5 Let ABC be an equilateral triangle whose sides have length 1. The midpoints of AB, BC are M, N respectively. Points K, L were chosen on AC so that $KLMN$ is a rectangle. Inside this rectangle are three semi-circles with the same radius, as in the picture (the endpoints are on the edges of the rectangle, and the arcs are tangent).
Find the minimum possible value of the radii of the semi-circles.

P6 Determine if there exists a set S of 5783 different real numbers with the following property:
For every $a, b\in S$ (not necessarily distinct) there are $c\neq d$ in S so that $a\cdot b=c+d$.

P7 Ana and Banana are playing a game. Initially, Ana secretly picks a number $1\leq A\leq 10^6$. In each subsequent turn of the game, Banana may pick a positive integer B, and Ana will reveal to him the most common digit in the product $A\cdot B$ (written in decimal notation). In the case when at least two digits are tied for being the most common, Ana will reveal all of them to Banana.

For example, if $A\cdot B=2022$, Ana will tell Banana that the digit 2 is the most common, while if $A\cdot B=5783783$, Ana will reveal that $3,7$ and 8 are the most common. Banana's goal is to determine with certainty the number A after some number of turns. Does he have a winning strategy?