European Mathematical Cup 2022

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– Junior

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1 Determine all positive integers n for which there exist positive divisors a, b, c of n such that $a > b > c$ and $a^2 - b^2$, $b^2 - c^2$, $a^2 - c^2$ are also divisors of n.

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2 Find all pairs (x, y) of positive real numbers such that xy is an integer and $x + y = \lfloor x^2 - y^2 \rfloor$.

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3 Let ABC be an acute-angled triangle with $AC > BC$, with incircle τ centered at I which touches BC and AC at points D and E, respectively. The point M on τ is such that $BM \parallel DE$ and M and B lie on the same halfplane with respect to the angle bisector of $\angle ACB$. Let F and H be the intersections of τ with BM and CM different from M, respectively. Let J be a point on the line AC such that $JM \parallel EH$. Let K be the intersection of JF and τ different from F. Prove that $ME \parallel KH$.

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4 A collection F of distinct (not necessarily non-empty) subsets of $X = \{1, 2, \ldots, 300\}$ is *lovely* if for any three (not necessarily distinct) sets A, B and C in F at most three out of the following eight sets are non-empty

$$A \cap B \cap C, \quad \overline{A} \cap B \cap C, \quad A \cap \overline{B} \cap C, \quad A \cap B \cap \overline{C},$$
$$\overline{A} \cap \overline{B} \cap C, \quad \overline{A} \cap B \cap \overline{C}, \quad A \cap \overline{B} \cap \overline{C}, \quad \overline{A} \cap \overline{B} \cap \overline{C}$$

where $\overline{S}$ denotes the set of all elements of X which are not in S.

What is the greatest possible number of sets in a lovely collection?

– Senior

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1 Let $n \geq 3$ be a positive integer. Alice and Bob are playing a game in which they take turns colouring the vertices of a regular n-gon. Alice plays the first move. Initially, no vertex is coloured. Both players start the game with 0 points.

In their turn, a player colours a vertex V which has not been coloured and gains k points where k is the number of already coloured neighbouring vertices of V. (Thus, k is either 0, 1 or 2.)

The game ends when all vertices have been coloured and the player with more points wins; if they have the same number of points, no one wins. Determine all $n \geq 3$ for which Alice has a winning strategy and all $n \geq 3$ for which Bob has a winning strategy.

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2 We say that a positive integer n is lovely if there exist a positive integer k and (not necessarily distinct) positive integers d_1, d_2, $\ldots$, d_k such that $n = d_1 d_2 \cdots d_k$ and $d_i^2 \mid n + d_i$ for $i = 1, 2, \ldots, k$.

a) Are there infinitely many lovely numbers?

b) Is there a lovely number, greater than 1, which is a perfect square of an integer?

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3 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^3) + f(y)^3 + f(z)^3 = 3xyz$$

for all real numbers x, y and z with $x + y + z = 0$.

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4 Five points A, B, C, D and E lie on a circle τ clockwise in that order such that $AB \parallel CE$ and $\angle ABC > 90^\circ$. Let k be a circle tangent to AD, CE and τ such that k and τ touch on the arc $\widehat{DE}$ not containing A, B and C. Let $F \neq A$ be the intersection of τ and the tangent line to k passing through A different from AD.

Prove that there exists a circle tangent to BD, BF, CE and τ.
