German National Olympiad 2002

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– Day 1

1 Find all real numbers a, b satisfying the following system of equations

$$2a^2 - 2ab + b^2 = a$$
$$4a^2 - 5ab + 2b^2 = b.$$

2 Minimal distance of a finite set of different points in space is length of the shortest segment, whose both ends belong to this set and segment has length greater than 0.
a) Prove there exist set of 8 points on sphere with radius R, whose minimal distance is greater than $1,15R$.
b) Does there exist set of 8 points on sphere with radius R, whose minimal distance is greater than $1,2R$?

3 Prove that for all primes p true is equality

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{4}$$

– Day 2

4 Given a positive real number a_1, we recursively define $a_{n+1} = 1 + a_1 a_2 \cdot \dots \cdot a_n$. Furthermore, let

$$b_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$

Prove that $b_n < \frac{2}{a_1}$ for all positive integers n and that this is the smallest possible bound.

5 Show that the triangle whose angles satisfy the equality

$$\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\cos^2 A + \cos^2 B + \cos^2 C} = 2$$

is right angled

6 Theo Travel, who has 5 children, has already visited 8 countries of the eurozone. From every country, he brought 5 not necessarily distinct coins home. Moreover, among these 40 coins there are exactly 5 of every value ($1, 2, 5, 10, 20,$ and 50 ct, 1 and 2 euro). He wants to give each child 8 coins such that they are from different countries and that each child gets the same amount of money. Is this always possible?