National Olympiad Second Round 2022

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–   Day 1

1   In triangle ABC, M is the midpoint of side BC, the bisector of angle BAC intersects BC and (ABC) at K and L, respectively. If the circle with diameter $[BC]$ is tangent to the external angle bisector of angle BAC, prove that this circle is tangent to (KLM) as well.

2   For positive integers k and n, we know $k \geq n!$. Prove that $\phi(k) \geq (n-1)!$

3   Let $a_1, a_2, \cdots, a_{2022}$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_{2022} = 1$. Find the maximum number of ordered pairs (i, j), $1 \leq i, j \leq 2022$, satisfying

$$a_i^2 + a_j \geq \frac{1}{2021}.$$

–   Day 2

4   For which real numbers a, there exist pairwise different real numbers x, y, z satisfying

$$\frac{x^3 + a}{y + z} = \frac{y^3 + a}{x + z} = \frac{z^3 + a}{x + y} = -3.$$

5   In triangle ABC, $90^o > \angle A > \angle B > \angle C$. Let the circumcenter and orthocenter of the triangle be O and H. OH intersects BC at T and the circumcenter of (AHO) is X. Prove that the reflection of H over XT lies on the circumcircle of triangle ABC.

6   In a school with 2022 students, either a museum trip or a nature trip is organized every day during a holiday. No student participates in the same type of trip twice, and the number of students attending each trip is different. If there are no two students participating in the same two trips together, find the maximum number of trips held.