**German National Olympiad 2003**

www.artofproblemsolving.com/community/c3236484

by sqrtX, parmenides51, Lukas8r20

– Day 1

**1** Solve the system of equations:

$$\begin{cases} x^3+y^3=7 \\ xy(x+y)=-2 \end{cases}$$

**2** There are four circles $k_1, k_2, k_3$ and $k_4$ of equal radius inside the triangle $ABC$. The circle $k_1$ touches the sides $AB, CA$ and the circle $k_4$, $k_2$ touches the sides $AB$, $BC$ and $k_4$, and $k_3$ touches the sides $AC, BC$ and $k_4$. Prove that the center of $k_4$ lies on the line connecting the incenter and circumcenter of $ABC$.

**3** Consider a $N \times N$ square board where $N \geq 3$ is an odd integer. The caterpillar Carl sits at the center of the square; all other cells contain distinct positive integers. An integer $n$ weights $1/n$ kilograms. Carl wants to leave the board but can eat at most $2$ kilograms. Determine whether Carl can always find a way out when

a) $N=2003$.

b) $N$ is an arbitrary odd integer.

– Day 2

**4** From the midpoints of the sides of an acute-angled triangle, perpendiculars are drawn to the adjacent sides. The resulting six straight lines bound the hexagon. Prove that its area is half the area of the original triangle.

**5** $n$ is a positive integer. Let $a(n)$ be the smallest number for which $n \mid a(n)!$

Find all solutions of:

$$\frac{a(n)}{n}=\frac{2}{3}$$

**6** Prove that there are infinitely many coprime, positive integers $a, b$ such that $a$ divides $b^2-5$ and $b$ divides $a^2-5$.