Final Round - Switzerland 2006

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– Day 1

1 Find all functions $f : R \to R$ such that for all $x, y \in R$ holds

$$yf(2x) - xf(2y) = 8xy(x^2 - y^2).$$

2 Let ABC be an equilateral triangle and let D be an inner point of the side BC. A circle is tangent to BC at D and intersects the sides AB and AC in the inner points M, N and P, Q respectively. Prove that $|BD| + |AM| + |AN| = |CD| + |AP| + |AQ|$.

3 Calculate the sum of digit of the number

$$9 \times 99 \times 9999 \times ... \times \underbrace{99...99}_{2^n}$$

where the number of nines doubles in each factor.

4 A circle with circumference 6n units is given and 3n points divide the circumference in n intervals of 1 unit, n intervals of 2 units, and n intervals of 3 units. Prove that there is at least one pair of points that are diametrically opposite to each other.

5 A circle k_1 lies within a second circle k_2 and touches it at point A. A line through A intersects k_1 again in B and k_2 in C. The tangent to k_1 through B intersects k_2 at points D and E. The tangents at k_1 passing through C intersects k_1 in points F and G. Prove that D, E, F and G lie on a circle.

– Day 2

6 At least three players have participated in a tennis tournament. Evey two players have played each other exactly once, and each player has at least one match won. Show that there are three players A, B, C such that A won against B, B won against C and C won against A.

7 Let $ABCD$ be a cyclic quadrilateral with $\angle ABC = 60^o$ and $|BC| = |CD|$. Prove that $|CD| + |DA| = |AB|$

8 People from n different countries sit at a round table. Assume that for every two members of the same country their neighbours sitting next to them on the right hand side are from different countries. Find the largest possible number of people sitting around the table?

9 Let a, b, c, d be real numbers. Prove that is

$$(a^2 + b^2 + 1)(c^2 + d^2 + 1) \geq 2(a + c)(b + d).$$

10 Decide whether there is an integer $n > 1$ with the following properties:
(a) n is not a prime number.
(b) For all integers a, $a^n - a$ is divisible by n