Final Round - Switzerland 2007

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– Day 1

1 Determine all positive real solutions of the following system of equations:

$$a = \ max\{\frac{1}{b},\frac{1}{c}\} \quad b = \max\{\frac{1}{c},\frac{1}{d}\} \quad c = \max\{\frac{1}{d},\frac{1}{e}\}$$

$$d = \max\{\frac{1}{e},\frac{1}{f}\} \quad e = \max\{\frac{1}{f},\frac{1}{a}\} \quad f = \max\{\frac{1}{a},\frac{1}{b}\}$$

2 Let a, b, c be three integers such that $a + b + c$ is divisible by 13. Prove that

$$a^{2007} + b^{2007} + c^{2007} + 2 \cdot 2007abc$$

is divisible by 13.

3 The plane is divided into unit squares. Each box should be be colored in one of n colors , so that if four squares can be covered with an L-tetromino, then these squares have four different colors (the L-Tetromino may be rotated and be mirrored). Find the smallest value of n for which this is possible.

4 Let ABC be an acute-angled triangle with $AB > AC$ and orthocenter H. Let D the projection of A on BC. Let E be the reflection of C wrt D. The lines AE and BH intersect at point S. Let N be the midpoint of AE and let M be the midpoint of BH. Prove that MN is perpendicular to DS.

5 Determine all functions $f : R_{\ge 0} \to R_{\ge 0}$ with the following properties:
(a) $f(1) = 0$,
(b) $f(x) > 0$ for all $x > 1$,
(c) For all $x, y \ge 0$ with $x + y > 0$ holds

$$f(xf(y))f(y) = f\left(\frac{xy}{x + y}\right)$$

– Day 2

6 Three equal circles k_1, k_2, k_3 intersect non-tangentially at a point P. Let A and B be the centers of circles k_1 and k_2. Let D and C be the intersection of k_3 with k_1 and k_2 respectively, which is different from P. Show that $ABCD$ is a parallelogram.

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7 Let a, b, c be nonnegative real numbers with arithmetic mean $m = \frac{a+b+c}{3}$. Provethat

$$\sqrt{a+\sqrt{b+\sqrt{c}}}+\sqrt{b+\sqrt{c+\sqrt{a}}}+\sqrt{c+\sqrt{a+\sqrt{b}}}\leq 3\sqrt{m+\sqrt{m+\sqrt{m}}}.$$

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8 Let $M \subset \{1, 2, 3, ..., 2007\}$ a set with the following property: Among every three numbers one can always choose two from M such that one is divisible by the other. How many numbers can M contain at most?

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9 Find all pairs (a, b) of natural numbers such that

$$\frac{a^3+1}{2ab^2+1}$$

is an integer.

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10 The plane is divided into equilateral triangles of side length 1. Consider a equilateral triangle of side length n whose sides lie on the grid lines. On every grid point on the edge and inside of this triangle lies a stone. In a move, a unit triangle is selected, which has exactly 2 corners with is covered with a stone. The two stones are removed, and the third corner is turned a new stone was laid. For which n is it possible that after finitely many moves only one stone left?
