2023 China National Olympiad

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– Day 1 (December 29th)

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1 Define the sequences $(a_n),(b_n)$ by

$$a_n,b_n>0,\forall n\in\mathbb{N_+}$$
$$a_{n+1}=a_n-\frac{1}{1+\sum_{i=1}^n\frac{1}{a_i}}$$
$$b_{n+1}=b_n+\frac{1}{1+\sum_{i=1}^n\frac{1}{b_i}}$$

1) If $a_{100}b_{100}=a_{101}b_{101}$, find the value of a_1-b_1;
2) If $a_{100}=b_{99}$, determine which is larger between $a_{100}+b_{100}$ and $a_{101}+b_{101}$.

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2 Let $\triangle ABC$ be an equilateral triangle of side length 1. Let D,E,F be points on BC,AC,AB respectively, such that $\frac{DE}{20} = \frac{EF}{22} = \frac{FD}{38}$. Let X,Y,Z be on lines BC,CA,AB respectively, such that $XY\perp DE, YZ\perp EF, ZX\perp FD$. Find all possible values of $\frac{1}{[DEF]} + \frac{1}{[XYZ]}$.

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3 Given positive integer m,n, color the points of the regular $(2m+2n)$-gon in black and white, $2m$ in black and $2n$ in white.
The *coloring distance* $d(B,C)$ of two black points B,C is defined as the smaller number of white points in the two paths linking the two black points.
The *coloring distance* $d(W,X)$ of two white points W,X is defined as the smaller number of black points in the two paths linking the two white points.
We define the matching of black points $\mathcal{B}$: label the $2m$ black points with $A_1,\cdots,A_m,B_1,\cdots,B_m$ satisfying no A_iB_i intersects inside the gon.
We define the matching of white points $\mathcal{W}$: label the $2n$ white points with $C_1,\cdots,C_n,D_1,\cdots,D_n$ satisfying no C_iD_i intersects inside the gon.
We define $P(\mathcal{B})=\sum_{i=1}^m d(A_i,B_i), P(\mathcal{W})=\sum_{j=1}^n d(C_j,D_j)$.
Prove that: $\max_{\mathcal{B}} P(\mathcal{B})=\max_{\mathcal{W}} P(\mathcal{W})$

– Day 2 (December 30th)

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4 Find the minimum positive integer $n\ge 3$, such that there exist n points $A_1,A_2,\cdots,A_n$ satisfying no three points are collinear and for any $1\le i\le n$, there exist $1\le j\le n (j\neq i)$, segment A_jA_{j+1} pass through the midpoint of segment A_iA_{i+1}, where $A_{n+1}=A_1$

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5 Prove that there exist $C > 0$, which satisfies the following conclusion:
For any infinite positive arithmetic integer sequence $a_1, a_2, a_3, \cdots$, if the greatest common divisor of a_1 and a_2 is squarefree, then there exists a positive integer $m \le C \cdot {a_2}^2$, such that a_m is squarefree.
Note: A positive integer N is squarefree if it is not divisible by any square number greater than 1.

Proposed by Qu Zhenhua

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6 There are $n(n \geq 8)$ airports, some of which have one-way direct routes between them. For any two airports a and b, there is at most one one-way direct route from a to b (there may be both one-way direct routes from a to b and from b to a). For any set A composed of airports $(1 \le |A| \le n-1)$, there are at least $4 \cdot \min\{|A|, n-|A|\}$ one-way direct routes from the airport in A to the airport not in A.
Prove that: For any airport x, we can start from x and return to the airport by no more than $\sqrt{2n}$ one-way direct routes.
