Switzerland - Final Round 2012

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– Day 1

1 There are 2012 chameleons sitting at a round table. At the beginning each has the color red or green. After every full minute, each chamaleon, which has two neighbors of the same color, changes its color from red to green or from green to red. All others keep their color. Show that after 2012 minutes there are at least 2 chameleons that have the same often changed color.

Es sitzen 2012 Chamaleons an einem runden Tisch. Am Anfang besitzt jedes die Farbe rot oder grun. Nach jeder vollen Minute wechselt jedes Cham aleon, welches zwei gleichfarbige Nachbarn hat, seine Farbe von rot zu grun respektive von gr un zu rot. Alle anderen behalten ihre Farbe. Zeige, dass es nach 2012 Minuten mindestens 2 Chamaleons gibt, welche gleich oft die Farbe gewechselt haben.

2 Determine all functions $f : R \to R$ such that for all $x, y \in R$ holds

$$f(f(x) + 2f(y)) = f(2x) + 8y + 6.$$

3 The circles k_1 and k_2 intersect at points D and P. The common tangent of the two circles on the side of D touches k_1 at A and k_2 at B. The straight line AD intersects k_2 for a second time at C. Let M be the center of the segment BC. Show that $\angle DPM = \angle BDC$.

4 Show that there is no infinite sequence of primes $p_1, p_2, p_3, ...$ there any for each k: $p_{k+1} = 2p_k - 1$ or $p_{k+1} = 2p_k + 1$ is fulfilled.

Note that not the same formula for every k.

5 Let n be a natural number. Let $A_1, A_2, ..., A_k$ be distinct 3-element subsets of $\{1, 2, ..., n\}$ such that $|A_i \cap A_j| \neq 1$ for all $1 \leq i, j \leq k$. Determine all n for which there are n such that these subsets exist.

Bestimme alle n, fur die es n solche Teilmengen gibt.

– Day 2

6 Let $ABCD$ be a parallelogram with at least an angle not equal to 90^o and k the circumcircle of the triangle ABC. Let E be the diametrically opposite point of B. Show that the circumcircle of the triangle ADE and k have the same radius.

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7 Let n and k be natural numbers such that $n = 3k + 2$. Show that the sum of all factors of n is divisible by 3.

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8 Consider a cube and two of its vertices A and B, which are the endpoints of a face diagonal. A *path* is a sequence of cube angles, each step of one angle along a cube edge is walked to one of the three adjacent angles. Let a be the number of paths of length 2012 that starts at point A and ends at A and let b be the number of ways of length 2012 that starts in A and ends in B. Decide which of the two numbers a and b is the larger.

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9 Let $a, b, c > 0$ be real numbers with $abc = 1$. Show

$$1 + ab + bc + ca \geq \min \left\{ \frac{(a+b)^2}{ab}, \frac{(b+c)^2}{bc}, \frac{(c+a)^2}{ca} \right\}.$$

When does equality holds?

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10 Let O be an inner point of an acute-angled triangle ABC. Let A_1, B_1 and C_1 be the projections of O on the sides BC, AC and AB respectively . Let P be the intersection of the perpendiculars on B_1C_1 and A_1C_1 from points$A$ and B respectilvey. Let H be the projection of P on AB. Show that points A_1, B_1, C_1 and H lie on a circle.

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