Switzerland - Final Round 2013

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–	Day 1
1	Find all triples (a, b, c) of natural numbers such that the sets $$\{gcd(a, b), gcd(b, c), gcd(c, a), lcm(a, b), lcm(b, c), lcm(c, a)\}$$ and $$\{2, 3, 5, 30, 60\}$$ are the same. Remark: For example, the sets $\{1, 2013\}$ and $\{1, 1, 2013\}$ are equal.
2	Let n be a natural number and $p_1, ..., p_n$ distinct prime numbers. Show that $$p_1^2 + p_2^2 + ... + p_n^2 > n^3$$
3	Let $ABCD$ be a cyclic quadrilateral with $\angle ADC = \angle DBA$. Furthermore, let E be the projection of A on BD. Show that $BC = DE - BE$.
4	Find all functions $f : R_{>0} \to R_{>0}$ with the following property: $$f\left(\frac{x}{y+1}\right) = 1 - xf(x + y)$$ for all $x > y > 0$.
5	Each of $2n + 1$ students chooses a finite, nonempty set of consecutive integers . Two students are friends if they have chosen a common number. Everyone student is friends with at least n other students. Show that there is a student who is friends with everyone else.
–	Day 2
6	There are two non-empty stacks of n and m coins on a table. The following operations are allowed: • The same number of coins are removed from both stacks. • The number of coins in a stack is tripled. For which pairs (n, m) is it possible that after finitely many operations, no coins are more available?

7 Let O be the center of the circle of the triangle ABC with $AB \neq AC$. Furthermore, let S and T be points on the rays AB and AC, such that $\angle ASO = \angle ACO$ and $\angle ATO = \angle ABO$. Show that ST bisects the segment BC.

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8 Let $a, b, c > 0$ be real numbers. Show the following inequality:

$$a^2 \cdot \frac{a-b}{a+b} + b^2 \cdot \frac{b-c}{b+c} + c^2 \cdot \frac{c-a}{c+a} \geq 0.$$

When does equality holds?

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9 Find all quadruples (p, q, m, n) of natural numbers such that p and q are prime and the the following equation is fulfilled:

$$p^m - q^3 = n^3$$

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10 Let $ABCD$ be a tangential quadrilateral with $BC > BA$. The point P is on the segment BC, such that $BP = BA$. Show that the bisector of $\angle BCD$, the perpendicular on line BC through P and the perpendicular on BD through A, intersect at one point.
