USA Team Selection Test for EGMO 2023

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Day 1 Thursday, December 8, 2022

1 There are 2022 equally spaced points on a circular track γ of circumference 2022. The points are labeled $A_1, A_2, \ldots, A_{2022}$ in some order, each label used once. Initially, Bunbun the Bunny begins at A_1. She hops along γ from A_1 to A_2, then from A_2 to A_3, until she reaches A_{2022}, after which she hops back to A_1. When hopping from P to Q, she always hops along the shorter of the two arcs $\widehat{PQ}$ of γ; if $\overline{PQ}$ is a diameter of γ, she moves along either semicircle.

Determine the maximal possible sum of the lengths of the 2022 arcs which Bunbun traveled, over all possible labellings of the 2022 points.

Kevin Cong

2 Consider pairs of functions (f, g) from the set of nonnegative integers to itself such that

- $f(0) + f(1) + f(2) + \cdots + f(42) \le 2022$;
- for any integers $a \ge b \ge 0$, we have $g(a+b) \le f(a) + f(b)$.

Determine the maximum possible value of $g(0) + g(1) + g(2) + \cdots + g(84)$ over all such pairs of functions.

Evan Chen (adapting from TST3, by Sean Li)

3 Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

Kevin Cong

Day 2 Thursday, January 12, 2023

4 Let ABC be a triangle with $AB + AC = 3BC$. The B-excircle touches side AC and line BC at E and D, respectively. The C-excircle touches side AB at F. Let lines CF and DE meet at P. Prove that $\angle PBC = 90^\circ$.

Ray Li

5 Let $\lfloor \bullet \rfloor$ denote the floor function. For nonnegative integers a and b, their *bitwise xor*, denoted

$a \oplus b$, is the unique nonnegative integer such that

$$\left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor - \left\lfloor \frac{a \oplus b}{2^k} \right\rfloor$$

is even for every $k \geq 0$. Find all positive integers a such that for any integers $x > y \geq 0$, we have

$$x \oplus ax \neq y \oplus ay.$$

Carl Schildkraut

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6 Let m and n be fixed positive integers. Tsvety and Freyja play a game on an infinite grid of unit square cells. Tsvety has secretly written a real number inside of each cell so that the sum of the numbers within every rectangle of size either m by n or n by m is zero. Freyja wants to learn all of these numbers.

One by one, Freyja asks Tsvety about some cell in the grid, and Tsvety truthfully reveals what number is written in it. Freyja wins if, at any point, Freyja can simultaneously deduce the number written in every cell of the entire infinite grid (If this never occurs, Freyja has lost the game and Tsvety wins).

In terms of m and n, find the smallest number of questions that Freyja must ask to win, or show that no finite number of questions suffice.

Nikolai Beluhov
