Argentina National Olympiad 2013

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–	Level 3
–	Day 1
1	On a table there are 2013 cards that have written, each one, a different integer number, from 1 to 2013; all the cards face down (you can't see what number they are). It is allowed to select any set of cards and ask if the average of the numbers written on those cards is integer. The answer will be true. a) Find all the numbers that can be determined with certainty by several of these questions. b) We want to divide the cards into groups such that the content of each group is known even though the individual value of each card in the group is not known. (For example, finding a group of 3 cards that contains $1, 2$, and 3, without knowing what number each card has.) What is the maximum number of groups that can be obtained?
2	In a convex quadrilateral $ABCD$ the angles $\angle A$ and $\angle C$ are equal and the bisector of $\angle B$ passes through the midpoint of the side CD. If it is known that $CD = 3AD$, calculate $\frac{AB}{BC}$.
3	Find how many are the numbers of 2013 digits $d_1d_2\ldots d_{2013}$ with odd digits $d_1, d_2, \ldots, d_{2013}$ such that the sum of 1809 terms $$d_1 \cdot d_2 + d_2 \cdot d_3 + \ldots + d_{1809} \cdot d_{1810}$$ has remainder 1 when divided by 4 and the sum of 203 terms $$d_{1810} \cdot d_{1811} + d_{1811} \cdot d_{1812} + \ldots + d_{2012} \cdot d_{2013}$$ has remainder 1 when dividing by 4.
–	Day 2
4	Let $x \geq 5, y \geq 6, z \geq 7$ such that $x^2 + y^2 + z^2 \geq 125$. Find the minimum value of $x + y + z$.
5	Given several nonnegative integers (repetitions allowed), the allowed operation is to choose a positive integer a and replace each number b greater than or equal to a by $b - a$ (the numbers a, if any, are replaced by 0). Initially, the integers from 1 are written on the blackboard until 2013 inclusive. After a few operations the numbers on the board have a sum equal to 10. Determine what the numbers that remained on the board could be. Find all the possibilities.

6 A positive integer n is called *pretty* if there exists two divisors d_1, d_2 of n $(1 \leq d_1, d_2 \leq n)$ such that $d_2 - d_1 = d$ for each divisor d of n (where $1 < d < n$).
Find the smallest pretty number larger than 401 that is a multiple of 401.