NIMO Summer Contest 2016

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by MSTang, djmathman

1 What is the value of
$$\left(9+\frac{9}{9}\right)^{9-9/9}-\frac{9}{9}?$$

Proposed by David Altizio

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2 Compute the number of permutations (a, b, c, x, y, z) of $(1, 2, 3, 4, 5, 6)$ which satisfy the five inequalities
$$a<b<c, \quad x<y<z, \quad a<x, \quad b<y, \quad \text{and} \quad c<z.$$

Proposed by Evan Chen

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3 Consider all 1001-element subsets of the set $\{1, 2, 3, ..., 2015\}$. From each such subset choose the median. Find the arithmetic mean of all these medians.

Proposed by Michael Ren

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4 Nine people sit in three rows of three chairs each. The probability that two of them, Celery and Drum, sit next to each other in the same row is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m+n$.

Proposed by Michael Tang

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5 Compute the number of non-empty subsets S of $\{-3, -2, -1, 0, 1, 2, 3\}$ with the following property: for any $k \geq 1$ distinct elements $a_1, \ldots, a_k \in S$ we have $a_1 + \cdots + a_k \neq 0$.

Proposed by Evan Chen

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6 A positive integer n is lucky if $2n+1$, $3n+1$, and $4n+1$ are all composite numbers. Compute the smallest lucky number.

Proposed by Michael Tang

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7 Suppose that a and b are real numbers such that $\sin(a)+\sin(b)=1$ and $\cos(a)+\cos(b)=\frac{3}{2}$. If the value of $\cos(a-b)$ can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, determine $100m+n$.

Proposed by Michael Ren

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8 Evan writes a computer program that randomly rearranges the digits $0, 2, 4, 6$, and 8 to create a five-digit number with no leading zeroes. If he executes this program once, the probability the program outputs an integer divisible by 4 can be written in the form $\frac{m}{n}$ where m and n are positive integers which share no common factors. What is $100m + n$?

Proposed by David Altizio

9 Compute the number of real numbers t such that

$$t = 50\sin(t - \lfloor t \rfloor).$$

Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Proposed by David Altizio

10 In rectangle $ABCD$, point M is the midpoint of AB and P is a point on side BC. The perpendicular bisector of MP intersects side DA at point X. Given that $AB = 33$ and $BC = 56$, find the least possible value of MX.

Proposed by Michael Tang

11 A set S of positive integers is *sum-complete* if there are positive integers m and n such that an integer a is the sum of the elements of some nonempty subset of S if and only if $m \le a \le n$.

Let S be a sum-complete set such that $\{1, 3\} \subset S$ and $|S| = 8$. Find the greatest possible value of the sum of the elements of S.

Proposed by Michael Tang

12 Let p be a prime. It is given that there exists a unique nonconstant function $\chi : \{1, 2, \ldots, p-1\} \to \{-1, 1\}$ such that $\chi(1) = 1$ and $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \not\equiv 0 \pmod p$ (here the product mn is taken mod p). For how many positive primes p less than 100 is it true that

$$\sum_{a=1}^{p-1} a^{\chi(a)} \equiv 0 \pmod p?$$

Here as usual a^{-1} denotes multiplicative inverse.

Proposed by David Altizio

13 The area of the region in the xy-plane satisfying the inequality

$$\min_{1 \le n \le 10} \max\left(\frac{x^2+y^2}{4n^2}, 2 - \frac{x^2+y^2}{4n^2-4n+1}\right) \le 1$$

is $k\pi$, for some integer k. Find k.

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Proposed by Michael Tang

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14 Find the smallest positive integer n such that $n^2 + 4$ has at least four distinct prime factors.

Proposed by Michael Tang

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15 Let ABC be a triangle with $AB = 17$ and $AC = 23$. Let G be the centroid of ABC, and let B_1 and C_1 be on the circumcircle of ABC with $BB_1 \parallel AC$ and $CC_1 \parallel AB$. Given that G lies on B_1C_1, the value of BC^2 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $100m + n$.

Proposed by Michael Ren

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