Dutch IMO TST 2016

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–	Day 1	
1	Let $\triangle ABC$ be a acute triangle. Let H the foot of the C-altitude in AB such that $AH = 3BH$, let M and N the midpoints of AB and AC and let P be a point such that $NP = NC$ and $CP = CB$ and B, P are located on different sides of the line AC. Prove that $\measuredangle APM = \measuredangle PBA$.	
2	In a $2^n \times 2^n$ square with n positive integer is covered with at least two non-overlapping rectangle pieces with integer dimensions and a power of two as surface. Prove that two rectangles of the covering have the same dimensions (Two rectangles have the same dimensions as they have the same width and the same height, wherein they, not allowed to be rotated.)	
3	Find all positive integers k for which the equation: $$\text{lcm}(m,n) - \text{gcd}(m,n) = k(m-n)$$ has no solution in integers positive (m,n) with $m \neq n$.	
4	Find all funtions $f : \mathbb{R} \to \mathbb{R}$ such that: $$f(xy-1) + f(x)f(y) = 2xy - 1$$ for all $x, y \in \mathbb{R}$.	
–	Day 2	
1	Prove that for all positive reals a, b, c we have: $a + \sqrt{ab} + \sqrt[3]{abc} \leq \frac{4}{3}(a+b+c)$	
2	Determine all pairs (a, b) of integers having the following property: there is an integer $d \geq 2$ such that $a^n + b^n + 1$ is divisible by d for all positive integers n.	
3	Let $\triangle\ ABC$ be an isosceles triangle with $\|AB\| = \|AC\|$. Let D, E and F be points on line segments BC, CA and AB, respectively, such that $\|BF\| = \|BE\|$ and such that ED is the internal angle bisector of $\angle BEC$. Prove that $\|BD\| = \|EF\|$ if and only if $\|AF\| = \|EC\|$.	
4	Determine the number of sets $A = \{a_1, a_2, ..., a_{1000}\}$ of positive integers satisfying $a_1 < a_2 < ... < a_{1000} \leq 2014$, for which we have that the set $S = \{a_i + a_j \mid 1 \leq i, j \leq 1000 \text{ with } i + j \in A\}$ is a subset of A.	

– Day 3

1 Let n be a positive integer. In a village, n boys and n girls are living.
For the yearly ball, n dancing couples need to be formed, each of which consists of one boy and one girl. Every girl submits a list, which consists of the name of the boy with whom she wants to dance the most, together with zero or more names of other boys with whom she wants to dance. It turns out that n dancing couples can be formed in such a way that every girl is paired with a boy who is on her list.
Show that it is possible to form n dancing couples in such a way that every girl is paired with a boy who is on her list, and at least one girl is paired with the boy with whom she wants to dance the most.

2 For distinct real numbers $a_1, a_2, ..., a_n$, we calculate the $\frac{n(n-1)}{2}$ sums $a_i + a_j$ with $1 \leq i < j \leq n$, and sort them in ascending order. Find all integers $n \geq 3$ for which there exist $a_1, a_2, ..., a_n$, for which this sequence of $\frac{n(n-1)}{2}$ sums form an arithmetic progression (i.e. the di erence between consecutive terms is constant).

3 Let k be a positive integer, and let $s(n)$ denote the sum of the digits of n.
Show that among the positive integers with k digits, there are as many numbers n satisfying $s(n) < s(2n)$ as there are numbers n satisfying $s(n) > s(2n)$.

4 Let Γ_1 be a circle with centre A and Γ_2 be a circle with centre B, with A lying on Γ_2. On Γ_2 there is a (variable) point P not lying on AB. A line through P is a tangent of Γ_1 at S, and it intersects Γ_2 again in Q, with P and Q lying on the same side of AB. A different line through Q is tangent to Γ_1 at T. Moreover, let M be the foot of the perpendicular to AB through P. Let N be the intersection of AQ and MT.
Show that N lies on a line independent of the position of P on Γ_2.