

International Olympiad of Metropolises

www.artofproblemsolving.com/community/c335778

by Wolowizard

– International Olympiad of Metropolises 2016

Day 1 September 6th

1 Find all positive integers n such that there exist n consecutive positive integers whose sum is a perfect square.

2 Let a_1, \dots, a_n be positive integers satisfying the inequality $\sum_{i=1}^n \frac{1}{a_i} \leq \frac{1}{2}$. Every year, the government of Optimistica publishes its Annual Report with n economic indicators. For each $i = 1, \dots, n$, the possible values of the i -th indicator are $1, 2, \dots, a_i$. The Annual Report is said to be optimistic if at least $n - 1$ indicators have higher values than in the previous report. Prove that the government can publish optimistic Annual Reports in an infinitely long sequence.

3 Let $A_1A_2\dots A_n$ be a cyclic convex polygon whose circumcenter is strictly in its interior. Let B_1, B_2, \dots, B_n be arbitrary points on the sides $A_1A_2, A_2A_3, \dots, A_nA_1$, respectively, other than the vertices. Prove that $\frac{B_1B_2}{A_1A_3} + \frac{B_2B_3}{A_2A_4} + \dots + \frac{B_nB_1}{A_nA_2} > 1$.

Day 2 September 7th

4 A convex quadrilateral $ABCD$ has right angles at A and C . A point E lies on the extension of the side AD beyond D so that $\angle ABE = \angle ADC$. The point K is symmetric to the point C with respect to point A . Prove that $\angle ADB = \angle AKE$.

5 Let $r(x)$ be a polynomial of odd degree with real coefficients. Prove that there exist only finitely many (or none at all) pairs of polynomials $p(x)$ and $q(x)$ with real coefficients satisfying the equation $(p(x))^3 + q(x^2) = r(x)$.

6 In a country with n cities, some pairs of cities are connected by one-way flights operated by one of two companies A and B . Two cities can be connected by more than one flight in either direction. An AB -word w is called implementable if there is a sequence of connected flights whose companies names form the word w . Given that every AB -word of length 2^n is implementable, prove that every finite AB -word is implementable. (An AB -word of length k is an arbitrary sequence of k letters A or B ; e.g. $AABA$ is a word of length 4.)