South African National Olympiad 2016

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1 At the start of the Mighty Mathematicians Football Team's first game of the season, their coach noticed that the jersey numbers of the 22 players on the field were all the numbers from 1 to 22. At halftime, the coach substituted her goal-keeper, with jersey number 1, for a reserve player. No other substitutions were made by either team at or before halftime. The coach noticed that after the substitution, no two players on the field had the same jersey number and that the sums of the jersey numbers of each of the teams were exactly equal. Determine
* the greatest possible jersey number of the reserve player,
* the smallest possible (positive) jersey number of the reserve player.

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2 Determine all pairs of real numbers a and b, $b > 0$, such that the solutions to the two equations

$$x^2 + ax + a = b \qquad \text{and} \qquad x^2 + ax + a = -b$$

are four consecutive integers.

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3 The inscribed circle of triangle ABC, with centre I, touches sides BC, CA and AB at D, E and F, respectively. Let P be a point, on the same side of FE as A, for which $\angle PFE = \angle BCA$ and $\angle PEF = \angle ABC$. Prove that P, I and D lie on a straight line.

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4 For which integers $n \geq 2$ is it possible to draw n straight lines in the plane in such a way that there are at least $n-2$ points where exactly three of the lines meet?

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5 For every positive integer n, determine the greatest possible value of the quotient

$$\frac{1 - x^n - (1-x)^n}{x(1-x)^n + (1-x)x^n}$$

where $0 < x < 1$.

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6 Let k and m be integers with $1 < k < m$. For a positive integer i, let L_i be the least common multiple of $1, 2, \ldots, i$.
Prove that k is a divisor of $L_i \cdot [\binom{m}{i} - \binom{m-k}{i}]$ for all $i \geq 1$. [Here, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ denotes a binomial coefficient. Note that $\binom{n}{i} = 0$ if $n < i$.]
