National Math Olympiad (3rd Round) 1996

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–	First Exam
1	Let a, b, c, d be positive real numbers. Prove that $$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c} \geq \frac{2}{3}.$$
2	Let $ABCD$ be a parallelogram. Construct the equilateral triangle DCE on the side DC and outside of parallelogram. Let P be an arbitrary point in plane of $ABCD$. Show that $$PA+PB+AD \geq PE.$$
3	Find all sets of real numbers $\{a_1, a_2, \ldots, _{1375}\}$ such that $$2\left(\sqrt{a_n-(n-1)}\right) \geq a_{n+1}-(n-1), \quad \forall n \in \{1, 2, \ldots, 1374\},$$ and $$2\left(\sqrt{a_{1375}-1374}\right) \geq a_1+1.$$
4	Show that there doesn't exist two infinite and separate sets A, B of points such that **(i)** There are no three collinear points in $A \cup B$, **(ii)** The distance between every two points in $A \cup B$ is at least 1, and **(iii)** There exists at least one point belonging to set B in interior of each triangle which all of its vertices are chosen from the set A, and there exists at least one point belonging to set A in interior of each triangle which all of its vertices are chosen from the set B.
–	Second Exam
1	Find all non-negative integer solutions of the equation $$2^x+3^y=z^2.$$

2 Let $ABCD$ be a convex quadrilateral. Construct the points $P, Q, R,$ and S on continue of $AB, BC, CD,$ and DA, respectively, such that

$$BP = CQ = DR = AS.$$

Show that if $PQRS$ is a square, then $ABCD$ is also a square.

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3 Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be n real numbers such that $a_1^k + a_2^k + \cdots + a_n^k \geq 0$ for all positive integers k. Suppose that $p = \max\{|a_1|, |a_2|, \dots, |a_n|\}$. Prove that $p = a_1$, and

$$(x - a_1)(x - a_2) \cdots (x - a_n) \leq x^n - a_1^n \qquad \forall x > a_1.$$

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4 Let n be a positive integer and suppose that $\phi(n) = \frac{n}{k}$, where k is the greatest perfect square such that $k \mid n$. Let $a_1, a_2, \dots, a_n$ be n positive integers such that $a_i = p_1^{a_{1i}} \cdot p_2^{a_{2i}} \cdots p_n^{a_{ni}}$, where p_i are prime numbers and a_{ji} are non-negative integers, $1 \leq i \leq n, 1 \leq j \leq n$. We know that $p_i \mid \phi(a_i)$, and if $p_i \mid \phi(a_j)$, then $p_j \mid \phi(a_i)$. Prove that there exist integers $k_1, k_2, \dots, k_m$ with $1 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq n$ such that

$$\phi(a_{k_1} \cdot a_{k_2} \cdots a_{k_m}) = p_1 \cdot p_2 \cdots p_n.$$

– Third Exam

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1 Suppose that S is a finite set of real numbers with the property that any two distinct elements of S form an arithmetic progression with another element in S. Give an example of such a set with 5 elements and show that no such set exists with more than 5 elements.

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2 Consider a semicircle of center O and diameter AB. A line intersects AB at M and the semicircle at C and D s.t. $MC > MD$ and $MB < MA$. The circumcircles od the AOC and BOD intersect again at K. Prove that $MK \perp KO$.

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3 Suppose that 10 points are given in the plane, such that among any five of them there are four lying on a circle. Find the minimum number of these points which must lie on a circle.

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4 Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0 - \{1\}$ such that

$$f(n+1) + f(n+3) = f(n+5)f(n+7) - 1375, \qquad \forall n \in \mathbb{N}.$$

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5 Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\measuredangle FHP = \measuredangle BAC$.

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6 Find all pairs (p, q) of prime numbers such that

$$m^{3pq} \equiv m \pmod{3pq} \qquad \forall m \in \mathbb{Z}.$$