National Math Olympiad (3rd Round) 1997

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– First Exam

1 Find all strictly ascending functions f such that for all $x \in \mathbb{R}$,

$$f(1-x) = 1 - f(f(x)).$$

2 Show that for any arbitrary triangle ABC, we have

$$\sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \sin\left(\frac{C}{2}\right) \leq \frac{abc}{(a+b)(b+c)(c+a)}.$$

3 There are 30 bags and there are 100 similar coins in each bag (coins in each bag are similar, coins of different bags can be different). The weight of each coin is an one digit number in grams. We have a digital scale which can weigh at most 999 grams in each weighing. Using this scale, we want to find the weight of coins of each bag.

(a) Show that this operation is possible by 10 times of weighing, and

(b) It's not possible by 9 times of weighing.

– Second Exam

1 Let P be a polynomial with integer coefficients. There exist integers a and b such that $P(a) \cdot P(b) = -(a-b)^2$. Prove that $P(a) + P(b) = 0$.

2 Let ABC and XYZ be two triangles. Define

$$A_1 = BC \cap ZX, A_2 = BC \cap XY,$$

$$B_1 = CA \cap XY, B_2 = CA \cap YZ,$$

$$C_1 = AB \cap YZ, C_2 = AB \cap ZX.$$

Hereby, the abbreviation $g \cap h$ means the point of intersection of two lines g and h.

Prove that $\frac{C_1C_2}{AB} = \frac{A_1A_2}{BC} = \frac{B_1B_2}{CA}$ holds if and only if $\frac{A_1C_2}{XZ} = \frac{C_1B_2}{ZY} = \frac{B_1A_2}{YX}$.

3 Let d be a real number such that $d^2 = r^2 + s^2$, where r and s are rational numbers. Prove that we can color all points of the plane with rational coordinates with two different colors such that the points with distance d have different colors.

– Third Exam

1 Suppose that a, b, x are positive integers such that

$$x^{a+b} = a^b b$$

Prove that $a = x$ and $b = x^x$.

2 In an acute triangle ABC, points D, E, F are the feet of the altitudes from A, B, C, respectively. A line through D parallel to EF meets AC at Q and AB at R. Lines BC and EF intersect at P. Prove that the circumcircle of triangle PQR passes through the midpoint of BC.

3 Let $S = \{x_0, x_1, \dots, x_n\}$ be a finite set of numbers in the interval $[0, 1]$ with $x_0 = 0$ and $x_1 = 1$. We consider pairwise distances between numbers in S. If every distance that appears, except the distance 1, occurs at least twice, prove that all the x_i are rational.

4 Let x, y, z be real numbers greater than 1 such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x+y+z}.$$

5 In an acute triangle ABC let AD and BE be altitudes, and AP and BQ be bisectors. Let I and O be centers of incircle and circumcircle, respectively. Prove that the points D, E, and I are collinear if and only if the points P, Q, and O are collinear.

6 Let $\mathcal{P}$ be the set of all points in $\mathbb{R}^n$ with rational coordinates. For the points $A, B \in \updownarrow P$, one can move from A to B if the distance AB is 1. Prove that every point in $\updownarrow P$ can be reached from any other point in $\mathcal{P}$ by a finite sequence of moves if and only if $n \geq 5$.