National Math Olympiad (3rd Round) 1998

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by Amir Hossein, moldovan, orl

– First Exam

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1 Define the sequence (x_n) by $x_0 = 0$ and for all $n \in \mathbb{N}$,

$$x_n = \begin{cases} x_{n-1} + (3^r - 1)/2, & \text{if } n = 3^{r-1}(3k+1); \\ x_{n-1} - (3^r + 1)/2, & \text{if } n = 3^{r-1}(3k+2). \end{cases}$$

where $k \in \mathbb{N}_0, r \in \mathbb{N}$. Prove that every integer occurs in this sequence exactly once.

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2 Let $ABCD$ be a convex pentagon such that

$$\angle DCB = \angle DEA = 90^\circ, \text{ and } DC = DE.$$

Let F be a point on AB such that $AF : BF = AE : BC$. Show that

$$\angle FEC = \angle BDC, \text{ and } \angle FCE = \angle ADE.$$

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3 Let A, B be two matrices with positive integer entries such that sum of entries of a row in A is equal to sum of entries of the same row in B and sum of entries of a column in A is equal to sum of entries of the same column in B. Show that there exists a sequence of matrices $A_1, A_2, A_3, \cdots, A_n$ such that all entries of the matrix A_i are positive integers and in the sequence

$$A = A_0, A_1, A_2, A_3, \cdots, A_n = B,$$

for each index i, there exist indexes k, j, m, n such that

$$A_{i+1} - A_i = \begin{array}{c c} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \end{array} \text{ or } \begin{array}{c c} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix} \end{array}.$$

That is, all indices of $A_{i+1} - A_i$ are zero, except the indices $(m, j), (m, k), (n, j)$, and (n, k).

– Second Exam

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1 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m, n,

(i) $mf(f(m)) = (f(m))^2$,
(ii) If $\gcd(m, n) = d$, then $f(mn) \cdot f(d) = d \cdot f(m) \cdot f(n)$,
(iii) $f(m) = m$ if and only if $m = 1$.

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2 Let $ABCD$ be a cyclic quadrilateral. Let E and F be variable points on the sides AB and CD, respectively, such that $AE : EB = CF : FD$. Let P be the point on the segment EF such that $PE : PF = AB : CD$. Prove that the ratio between the areas of triangles APD and BPC does not depend on the choice of E and F.

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3 Let $n(r)$ be the maximum possible number of points with integer coordinates on a circle with radius r in Cartesian plane. Prove that $n(r) < 6\sqrt[3]{3\pi r^2}$.

– Third Exam

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1 A one-player game is played on a $m \times n$ table with $m \times n$ nuts. One of the nuts' sides is black, and the other side of them is white. In the beginning of the game, there is one nut in each cell of the table and all nuts have their white side upwards except one cell in one corner of the table which has the black side upwards. In each move, we should remove a nut which has its black side upwards from the table and reverse all nuts in adjacent cells (i.e. the cells which share a common side with the removed nut's cell). Find all pairs (m, n) for which we can remove all nuts from the table.

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2 Let M and N be two points inside triangle ABC such that

$$\angle MAB = \angle NAC \quad \text{and} \quad \angle MBA = \angle NBC.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

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3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, y,

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y.$$

– Fourth Exam

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1 Determine all positive integers n for which there exists an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.

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2 Let $ABCDEF$ be a convex hexagon such that $AB = BC, CD = DE$ and $EF = FA$. Prove that

$$\frac{AB}{BE} + \frac{CD}{AD} + \frac{EF}{CF} \geq \frac{3}{2}.$$

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3 Let ABC be a given triangle. Consider any painting of points of the plane in red and green. Show that there exist either two red points on the distance 1, or three green points forming a triangle congruent to triangle ABC.

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4 Let be given $r_1, r_2, \ldots, r_n \in \mathbb{R}$. Show that there exists a subset I of $\{1, 2, \ldots, n\}$ which which has one or two elements in common with the sets $\{i, i+1, i+2\}, (1 \leq i \leq n-2)$ such that

$$\left| \sum_{i \in I} r_i \right| \geqslant \frac{1}{6} \sum_{i=1}^{n} |r_i| \, .$$

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5 In a triangle ABC, the bisector of angle BAC intersects BC at D. The circle Γ through A which is tangent to BC at D meets AC again at M. Line BM meets Γ again at P. Prove that line AP is a median of $\triangle ABD$.

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6 For any two nonnegative integers n and k satisfying $n \geq k$, we define the number $c(n,k)$ as follows:

- $c\left(n,0\right) = c\left(n,n\right) = 1$ for all $n \geq 0$;

- $c\left(n+1,k\right) = 2^{k}c\left(n,k\right) + c\left(n,k-1\right)$ for $n \geq k \geq 1$.

Prove that $c\left(n,k\right) = c\left(n,n-k\right)$ for all $n \geq k \geq 0$.
