**National Math Olympiad (3rd Round) 2000**

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– 2nd round

**Day 1**

**1** Does there exist a natural number $N$ which is a power of $2$, such that one can permute its decimal digits to obtain a different power of $2$?

**2** Call two circles in three-dimensional space pairwise tangent at a point $P$ if they both pass through $P$ and lines tangent to each circle at $P$ coincide. Three circles not all lying in a plane are pairwise tangent at three distinct points. Prove that there exists a sphere which passes through the three circles.

**3** In a deck of $n > 1$ cards, some digits from $1$ to $8$ are written on each card. A digit may occur more than once, but at most once on a certain card. On each card at least one digit is written, and no two cards are denoted by the same set of digits. Suppose that for every $k = 1, 2, \dots, 7$ digits, the number of cards that contain at least one of them is even. Find $n$.

**Day 2**

**1** A sequence of natural numbers $c_1, c_2, \dots$ is called *perfect* if every natural number $m$ with $1 \le m \le c_1 + \dots + c_n$ can be represented as $m = \frac{c_1}{a_1} + \frac{c_2}{a_2} + \dots + \frac{c_n}{a_n}$ Given $n$, find the maximum possible value of $c_n$ in a perfect sequence $(c_i)$.

**2** Circles $C_1$ and $C_2$ with centers at $O_1$ and $O_2$ respectively meet at points $A$ and $B$. The radii $O_1B$ and $O_2B$ meet $C_1$ and $C_2$ at $F$ and $E$. The line through $B$ parallel to $EF$ intersects $C_1$ again at $M$ and $C_2$ again at $N$. Prove that $MN = AE + AF$.

**3** Two triangles $ABC$ and $A'B'C'$ are positioned in the space such that the length of every side of $\triangle ABC$ is not less than $a$, and the length of every side of $\triangle A'B'C'$ is not less than $a'$. Prove that one can select a vertex of $\triangle ABC$ and a vertex of $\triangle A'B'C'$ so that the distance between the two selected vertices is not less than $\sqrt{\frac{a^2+a'^2}{3}}$.

– 3rd round

**Day 1**

**1** Two circles intersect at two points $A$ and $B$. A line $\ell$ which passes through the point $A$ meets the two circles again at the points $C$ and $D$, respectively. Let $M$ and $N$ be the midpoints of the arcs $BC$ and $BD$ (which do not contain the point $A$) on the respective circles. Let $K$ be the midpoint of the segment $CD$. Prove that $\measuredangle MKN = 90^\circ$.

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**2** Let $A$ and $B$ be arbitrary finite sets and let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be functions such that $g$ is not onto. Prove that there is a subset $S$ of $A$ such that $\frac{A}{S} = g(\frac{B}{f(S)})$.

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**3** Suppose $f : \mathbb{N} \longrightarrow \mathbb{N}$ is a function that satisfies $f(1) = 1$ and $f(n+1) = \{ \begin{array}{ll} f(n)+2 & \text{if } n = f(f(n)-n+1), \\ f(n)+1 & \text{Otherwise} \end{array}$

$(a)$ Prove that $f(f(n)-n+1)$ is either $n$ or $n+1$. $(b)$ Determine $f$.

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**Day 2**

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**1** Let us denote $\prod = \{(x,y)|y>0\}$. We call a *semicircle* in $\prod$ with center on the $x-$ axis a *semi-line*. Two intersecting *semi-lines* determine four *semi-angles*. A bisector of a *semi-angle* is a *semi-line* that bisects the *semi-angle*. Prove that in every *semi-triangle* (determined by three *semi-lines*) the bisectors are concurrent.

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**2** Find all f:N $\longrightarrow$ N that:
**a)** $f(m) = 1 \iff m = 1$
**b)** $d = gcd(m,n) f(m \cdot n) = \frac{f(m) \cdot f(n)}{f(d)}$
**c)** $f^{2000}(m) = f(m)$

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**3** Let $n$ points be given on a circle, and let $nk+1$ chords between these points be drawn, where $2k+1 < n$. Show that it is possible to select $k+1$ of the chords so that no two of them intersect.

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**–** 4th round

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**Day 1**

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**1** In a tennis tournament where $n$ players $A_1, A_2, \dots, A_n$ take part, any two players play at most one match, and $k \leq \frac{n(n-1)}{2}$ 2 matches are played. The winner of a match gets $1$ point while the loser gets $0$. Prove that a sequence $d_1, d_2, \dots, d_n$ of nonnegative integers can be the sequence of scores of the players ($d_i$ being the score of$A_i$) if and only if $(i)$ $d_1 + d_2 + \cdots + d_n = k$, and $(ii)$ for any$X \subset \{A_1, \dots, A_n\}$, the number of matches between the players in $X$ is at most $\sum_{A_j \in X} d_j$

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**2** Isosceles triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed on the sides of a triangle $A_1A_2A_3$ as the bases, outside the triangle. Let $O_1$ be a point

outside $\Delta A_1A_2A_3$ such that $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$ and $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$.
Prove that $A_1O_1 \perp O_2O_3$, and if $T$ is the projection of $O_1$ onto $A_2A_3$,
then $\frac{A_1O_1}{O_2O_3} = 2\frac{O_1T}{A_2A_3}$.

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**3** A circle$\Gamma$ with radius $R$ and center $\omega$, and a line $d$ are drawn on a plane,
such that the distance of $\omega$ from $d$ is greater than $R$. Two points $M$ and $N$ vary on $d$ so that the circle with diameter $MN$ is tangent to $\Gamma$. Prove
that there is a point $P$ in the plane from which all the segments $MN$ are
visible at a constant angle.

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**Day 2**

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**1** Let $n$ be a positive integer. Suppose $S$ is a set of ordered $n-$ tuples of
nonnegative integers such that, whenever $(a_1, \dots, an) \in S$ and $b_i$ are nonnegative integers with$b_i \le a_i$, the $n-$ tuple $(b_1, \dots, b_n)$ is also in $S$. If $h_m$
is the number of elements of $S$ with the sum of components equal to$m$,
prove that $h_m$ is a polynomial in $m$ for all sufficiently large$m$.

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**2** Suppose that $a, b, c$ are real numbers such that for all positive numbers $x_1, x_2, \dots, x_n$ we have
$(\frac{1}{n}\sum_{i=1}^n x_i)^a(\frac{1}{n}\sum_{i=1}^n x_i^2)^b(\frac{1}{n}\sum_{i=1}^n x_i^3)^c \ge 1$

Prove that vector $(a, b, c)$ is a nonnegative linear combination of vectors $(-2, 1, 0)$ and $(-1, 2, -1)$.

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**3** Prove that for every natural number $n$ there exists a polynomial $p(x)$ with
integer coefficients such that$p(1), p(2), ..., p(n)$ are distinct powers of $2$ .

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