National Math Olympiad (3rd Round) 2002

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1 Let $a,b,c\in\mathbb{R}^n, a+b+c=0$ and $\lambda>0$. Prove that

$$\prod_{cycle}\frac{|a|+|b|+(2\lambda+1)|c|}{|a|+|b|+|c|}\geq(2\lambda+3)^3$$

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2 $f:\mathbb{R}\longrightarrow\mathbb{R}^+$ is a non-decreasing function. Prove that there is a point $a\in\mathbb{R}$ that

$$f(a+\frac{1}{f(a)})<2f(a)$$

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3 a_n is a sequence that $a_1=1,a_2=2,a_3=3$, and

$$a_{n+1}=a_n-a_{n-1}+\frac{a_n^2}{a_{n-2}}$$

Prove that for each natural n, a_n is integer.

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4 a_n (n is integer) is a sequence from positive reals that

$$a_n\geq\frac{a_{n+2}+a_{n+1}+a_{n-1}+a_{n-2}}{4}$$

Prove a_n is constant.

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5 ω is circumcirlce of triangle ABC. We draw a line parallel to BC that intersects AB,AC at E,F and intersects ω at U,V. Assume that M is midpoint of BC. Let ω' be circumcircle of UMV. We know that $R(ABC)=R(UMV)$. ME and ω' intersect at T, and FT intersects ω' at S. Prove that EF is tangent to circumcircle of MCS.

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6 M is midpoint of $BC.P$ is an arbitary point on BC. C_1 is tangent to big circle.Suppose radius of C_1 is r_1
Radius of C_4 is equal to radius of C_1 and C_4 is tangent to BC at P. C_2 and C_3 are tangent to big circle and line BC and circle C_4.
`http://aycu01.webshots.com/image/4120/2005120338156776027_rs.jpg`
Prove :

$$r_1+r_2+r_3=R$$

(R radius of big circle)

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7 In triangle ABC, AD is angle bisector (D is on BC) if $AB+AD=CD$ and $AC+AD=BC$, what are the angles of ABC?

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8 Circles C_1 and C_2 are tangent to each other at K and are tangent to circle C at M and N. External tangent of C_1 and C_2 intersect C at A and B. AK and BK intersect with circle C at E and F respectively. If AB is diameter of C, prove that EF and MN and OK are concurrent. (O is center of circle C.)

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9 Let M and N be points on the side BC of triangle ABC, with the point M lying on the segment BN, such that $BM=CN$. Let P and Q be points on the segments AN and AM, respectively, such that $\measuredangle PMC=\measuredangle MAB$ and $\measuredangle QNB=\measuredangle NAC$. Prove that $\measuredangle QBC=\measuredangle PCB$.

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10 H,I,O,N are orthogonal center, incenter, circumcenter, and Nagelian point of triangle ABC. I_a,I_b,I_c are excenters of ABC corresponding vertices A,B,C. S is point that O is midpoint of HS. Prove that centroid of triangles $I_aI_bI_c$ and SIN concide.

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11 In an $m\times n$ table there is a policeman in cell $(1,1)$, and there is a thief in cell (i,j). A move is going from a cell to a neighbor (each cell has at most four neighbors). Thief makes the first move, then the policeman moves and ... For which (i,j) the policeman can catch the thief?

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12 We have a bipartite graph G (with parts X and Y). We orient each edge arbitrarily. *Hessam* chooses a vertex at each turn and reverse the orientation of all edges that v is one of their endpoint. Prove that with these steps we can reach to a graph that for each vertex v in part X, $\deg^{+}(v)\geq\deg^{-}(v)$ and for each vertex in part Y, $\deg^{+}v\leq\deg^{-}v$

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13 f,g are two permutations of set $X=\{1,\dots,n\}$. We say f,g have common points iff there is a $k\in X$ that $f(k)=g(k)$.
a) If $m>\frac{n}{2}$, prove that there are m permutations $f_1,f_2,\dots,f_m$ from X that for each permutation $f\in X$, there is an index i that f,f_i have common points.
b) Prove that if $m\leq\frac{n}{2}$, we can not find permutations $f_1,f_2,\dots,f_m$ satisfying the above condition.

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14 A subset S of $\mathbb{N}$ is *eventually linear* iff there are $k,N\in\mathbb{N}$ that for $n>N, n\in S\Longleftrightarrow k|n$. Let S be a subset of $\mathbb{N}$ that is closed under addition. Prove that S is eventually linear.

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15 Let A be be a point outside the circle C, and AB and AC be the two tangents from A to this circle C. Let L be an arbitrary tangent to C that cuts AB and AC in P and Q. A line through P parallel to AC cuts BC in R. Prove that while L varies, QR passes through a fixed point. :)

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16 For positive a,b,c,

$$a^2+b^2+c^2+abc=4$$

Prove $a+b+c\leq 3$

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17 Find the smallest natural number n that the following statement holds :
Let A be a finite subset of $\mathbb{R}^2$. For each n points in A there are two lines including these n points. All of the points lie on two lines.

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18 Find all continious $f : \mathbb{R} \longrightarrow \mathbb{R}$ that for any x, y

$$f(x) + f(y) + f(xy) = f(x + y + xy)$$

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19 I is incenter of triangle ABC. Incircle of ABC touches AB, AC at X, Y. XI intersects incircle at M. Let $CM \cap AB = X'$. L is a point on the segment $X'C$ that $X'L = CM$. Prove that A, L, I are collinear iff $AB = AC$.

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20 $a_0 = 2, a_1 = 1$ and for $n \geq 1$ we know that : $a_{n+1} = a_n + a_{n-1}$ m is an even number and p is prime number such that p divides $a_m - 2$. Prove that p divides $a_{m+1} - 1$.

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21 Excircle of triangle ABC corresponding vertex A, is tangent to BC at P. AP intersects circumcircle of ABC at D. Prove

$$r(PCD) = r(PBD)$$

whcih $r(PCD)$ and $r(PBD)$ are inradii of triangles PCD and PBD.

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22 15000 years ago Tilif ministry in Persia decided to define a code for $n \geq 2$ cities. Each code is a sequence of $0, 1$ such that no code start with another code. We know that from 2^m calls from foreign countries to Persia 2^{m-a_i} of them where from the i-th city (So $\sum_{i=1}^{n} \frac{1}{2^{a_i}} = 1$). Let l_i be length of code assigned to i-th city. Prove that $\sum_{i=1}^{n} \frac{l_i}{2^i}$ is minimum iff $\forall i,\ l_i = a_i$

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23 Find all polynomials p with real coefficients that if for a real a,$p(a)$ is integer then a is integer.

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24 A, B, C are on circle $\mathcal{C}$. I is incenter of ABC , D is midpoint of arc BAC. W is a circle that is tangent to AB and AC and tangent to $\mathcal{C}$ at P. (W is in $\mathcal{C}$)
Prove that P and I and D are on a line.

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25 An ant walks on the interior surface of a cube, he moves on a straight line. If ant reaches to an edge the he moves on a straight line on cube's net. Also if he reaches to a vertex he will return his path.
a) Prove that for each beginning point ant can has infinitely many choices for his direction that its path becomes periodic.
b) Prove that if if the ant starts from point A and its path is periodic, then for each point B if ant starts with this direction, then his path becomes periodic.
