National Math Olympiad (3rd Round) 2004

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1 We say $m \circ n$ for natural m,n $\Longleftrightarrow$
nth number of binary representation of m is 1 or mth number of binary representation of n is 1.
and we say $m \bullet n$ if and only if m, n doesn't have the relation $\circ$
We say $A \subset \mathbb{N}$ is golden $\Longleftrightarrow$ $\forall U, V \subset A$ that are finite and arenot empty and $U \cap V = \emptyset$,There exist $z \in A$ that $\forall x \in U, y \in V$ we have $z \circ x, z \bullet y$
Suppose $\mathbb{P}$ is set of prime numbers.Prove if $\mathbb{P} = P_1 \cup ... \cup P_k$ and $P_i \cap P_j = \emptyset$ then one of $P_1, ..., P_k$ is golden.

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2 A is a compact convex set in plane. Prove that there exists a point $O \in A$, such that for every line XX' passing through O, where X and X' are boundary points of A, then

$$\frac{1}{2} \leq \frac{OX}{OX'} \leq 2.$$

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3 Suppose $V = \mathbb{Z}_2^n$ and for a vector $x = (x_1, ..x_n)$ in V and permutation σ.We have $x_\sigma = (x_{\sigma(1)}, ..., x_{\sigma(n)})$
Suppose $n = 4k + 2, 4k + 3$ and $f : V \to V$ is injective and if x and y differ in more than $n/2$ places then $f(x)$ and $f(y)$ differ in more than $n/2$ places.
Prove there exist permutaion σ and vector v that $f(x) = x_\sigma + v$

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4 We have finite white and finite black points that for each 4 oints there is a line that white points and black points are at different sides of this line.Prove there is a line that all white points and black points are at different side of this line.

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5 assume that k,n are two positive integer $k \leq n$count the number of permutation $\{1, \dots, n\}$ st for any $1 \leq i, j \leq k$and any positive integer m we have $f^m(i) \neq j$ (f^m meas iterarte function,)

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6 assume that we have a n*n table we fill it with 1,...,n such that each number exists exactly n times prove that there exist a row or column such that at least $\sqrt{n}$ diffrent number are contained.

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7 Suppose F is a polygon with lattice vertices and sides parralell to x-axis and y-axis.Suppose $S(F), P(F)$ are area and perimeter of F.
Find the smallest k that: $S(F) \leq k.P(F)^2$

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8 $\mathbb{P}$ is a n-gon with sides $l_1, ..., l_n$ and vertices on a circle. Prove that no n-gon with this sides has area more than $\mathbb{P}$

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9 Let ABC be a triangle, and O the center of its circumcircle.
Let a line through the point O intersect the lines AB and AC at the points M and N, respectively. Denote by S and R the midpoints of the segments BN and CM, respectively.
Prove that $\measuredangle ROS = \measuredangle BAC$.

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10 $f : \mathbb{R}^2 \to \mathbb{R}^2$ is injective and surjective. Distance of X and Y is not less than distance of $f(X)$ and $f(Y)$. Prove for A in plane:

$$S(A) \geq S(f(A))$$

where $S(A)$ is area of A

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11 assume that ABC is acute traingle and AA' is median we extend it until it meets circumcircle at A". let AP_a be a diameter of the circumcircle. the pependicular from A' to AP_a meets the tangent to circumcircle at A" in the point X_a; we define X_b, X_c similary . prove that X_a, X_b, X_c are one a line.

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12 $\mathbb{N}_{10}$ is generalization of $\mathbb{N}$ that every hypernumber in $\mathbb{N}_{10}$ is something like: $\overline{...a_2a_1a_0}$ with $a_i \in 0, 1..9$
(Notice that $\overline{...000} \in \mathbb{N}_{10}$)
Also we easily have $+, *$ in $\mathbb{N}_{10}$.
first k number of $a * b$= first k nubmer of (first k number of a * first k number of b)
first k number of $a + b$= first k nubmer of (first k number of a + first k number of b)
Fore example $\overline{...999} + \overline{...0001} = \overline{...000}$
Prove that every monic polynomial in $\mathbb{N}_{10}[x]$ with degree d has at most d^2 roots.

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13 Suppose f is a polynomial in $\mathbb{Z}[X]$ and m is integer .Consider the sequence a_i like this $a_1 = m$ and $a_{i+1} = f(a_i)$ find all polynomials f and alll integers m that for each i:

$$a_i | a_{i+1}$$

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14 We define $f : \mathbb{N} \to \mathbb{N}$, $f(n) = \sum_{k=1}^{n} (k, n)$.

a) Show that if $\gcd(m, n) = 1$ then we have $f(mn) = f(m) \cdot f(n)$;

b) Show that $\sum_{d|n} f(d) = nd(n)$.

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15 This problem is easy but nobody solved it.
point A moves in a line with speed v and B moves also with speed v' that at every time the direction of move of B goes from A.We know $v \geq v'$.If we know the point of beginning of path of A, then B must be where at first that B can catch A.

16 Let ABC be a triangle . Let point X be in the triangle and AX intersects BC in Y . Draw the perpendiculars YP, YQ, YR, YS to lines CA, CX, BX, BA respectively. Find the necessary and sufficient condition for X such that $PQRS$ be cyclic .

17 Let $p = 4k+1$ be a prime. Prove that p has at least $\frac{\phi(p-1)}{2}$ primitive roots.

18 Prove that for any n, there is a subset $\{a_1, \dots, a_n\}$ of $\mathbb{N}$ such that for each subset S of $\{1, \dots, n\}$, $\sum_{i \in S} a_i$ has the same set of prime divisors.

19 Find all integer solutions of $p^3 = p^2 + q^2 + r^2$ where p, q, r are primes.

20 $p(x)$ is a polynomial in $\mathbb{Z}[x]$ such that for each $m, n \in \mathbb{N}$ there is an integer a such that $n \mid p(a^m)$. Prove that 0 or 1 is a root of $p(x)$.

21 $a_1, a_2, \dots, a_n$ are integers, not all equal. Prove that there exist infinitely many prime numbers p such that for some k

$$p \mid a_1^k + \cdots + a_n^k.$$

22 Suppose that $\mathcal{F}$ is a family of subsets of X. A, B are two subsets of X s.t. each element of $\mathcal{F}$ has non-empty intersection with A, B. We know that no subset of X with $n-1$ elements has this property. Prove that there is a representation A, B in the form $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, such that for each $1 \le i \le n$, there is an element of $\mathcal{F}$ containing both a_i, b_i.

23 $\mathcal{F}$ is a family of 3-subsets of set X. Every two distinct elements of X are exactly in k elements of $\mathcal{F}$. It is known that there is a partition of $\mathcal{F}$ to sets X_1, X_2 such that each element of $\mathcal{F}$ has non-empty intersection with both X_1, X_2. Prove that $|X| \le 4$.

24 In triangle ABC, points M, N lie on line AC such that $MA = AB$ and $NB = NC$. Also K, L lie on line BC such that $KA = KB$ and $LA = LC$. It is know that $KL = \frac{1}{2}BC$ and $MN = AC$. Find angles of triangle ABC.

25 Finitely many convex subsets of $\mathbb{R}^3$ are given, such that every three have non-empty intersection. Prove that there exists a line in $\mathbb{R}^3$ that intersects all of these subsets.

26 Finitely many points are given on the surface of a sphere, such that every four of them lie on the surface of open hemisphere. Prove that all points lie on the surface of an open hemisphere.

27 $\Delta_1, \dots, \Delta_n$ are n concurrent segments (their lines concur) in the real plane. Prove that if for every three of them there is a line intersecting these three segments, then there is a line that intersects all of the segments.

28 Find all prime numbers p such that $p=m^2+n^2$ and $p\mid m^3+n^3-4$.

29 Incircle of triangle ABC touches AB,AC at P,Q. BI,CI intersect with PQ at K,L. Prove that circumcircle of ILK is tangent to incircle of ABC if and only if $AB+AC=3BC$.

30 Find all polynomials $p\in\mathbb{Z}[x]$ such that $(m,n)=1\Rightarrow(p(m),p(n))=1$