National Math Olympiad (3rd Round) 2005
www.artofproblemsolving.com/community/c3492
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Day 1

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1 Suppose $a, b, c \in \mathbb{R}^+$. Prove that :

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

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2 Suppose $\{x_n\}$ is a decreasing sequence that $\lim_{n\to\infty} x_n = 0$. Prove that $\sum(-1)^n x_n$ is convergent

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3 Find all $\alpha > 0$ and $\beta > 0$ that for each $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n) \in \mathbb{R^+}^n$ that:

$$(\sum x_i^\alpha)(\sum y_i^\beta) \geq \sum x_i y_i$$

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4 Suppose $P, Q \in \mathbb{R}[x]$ that $deg\ P = deg\ Q$ and $PQ' - QP'$ has no real root. Prove that for each $\lambda \in \mathbb{R}$ number of real roots of P and $\lambda P + (1-\lambda)Q$ are equal.

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5 Suppose $a, b, c \in \mathbb{R}^+$and

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2$$

Prove that $ab + ac + bc \leq \frac{3}{2}$

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6 Suppose $A \subseteq \mathbb{R}^m$ is closed and non-empty. Let $f : A \to A$ is a lipchitz function with constant less than 1. (ie there exist $c < 1$ that $|f(x) - f(y)| < c|x - y|,\ \forall x, y \in A$). Prove that there exists a unique point $x \in A$ such that $f(x) = x$.

Day 2

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1 From each vertex of triangle ABC we draw 3 arbitary parrallell lines, and from each vertex we draw a perpendicular to these lines. There are 3 rectangles that one of their diagnals is triangle's side. We draw their other diagnals and call them ℓ_1, ℓ_2 and ℓ_3.

a) Prove that ℓ_1, ℓ_2 and ℓ_3 are concurrent at a point P.

b) Find the locus of P as we move the 3 arbitary lines.

2 Suppose O is circumcenter of triangle ABC. Suppose $\frac{S(OAB)+S(OAC)}{2} = S(OBC)$. Prove that the distance of O (circumcenter) from the radical axis of the circumcircle and the 9-point circle is

$$\frac{a^2}{\sqrt{9R^2-(a^2+b^2+c^2)}}$$

3 Prove that in acute-angled traingle ABC if r is inradius and R is radius of circumcircle then:

$$a^2+b^2+c^2 \geq 4(R+r)^2$$

4 Suppose in triangle ABC incircle touches the side BC at P and $\angle APB = \alpha$. Prove that :

$$\frac{1}{p-b}+\frac{1}{p-c}=\frac{2}{rtg\alpha}$$

5 Suppose H and O are orthocenter and circumcenter of triangle ABC. ω is circumcircle of ABC. AO intersects with ω at A_1. A_1H intersects with ω at A' and A'' is the intersection point of ω and AH. We define points $B',\ B'',\ C'$ and C'' similiarly. Prove that $A'A'', B'B''$ and $C'C''$ are concurrent in a point on the Euler line of triangle ABC.

Day 3

1 Find all $n,p,q \in \mathbb{N}$ that:

$$2^n+n^2=3^p7^q$$

2 Let $a \in \mathbb{N}$ and $m=a^2+a+1$. Find the number of $0 \leq x \leq m$ that:

$$x^3 \equiv 1 (\text{mod } m)$$

3 $p(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ that deg p is odd. $q(x), r(x)$ are polynomials with rational coefficients that $p(x)|q(x)^2+q(x).r(x)+r(x)^2$. Prove that

$$p(x)^2|q(x)^2+q(x).r(x)+r(x)^2$$

4 k is an integer. We define the sequence $\{a_n\}_{n=0}^{\infty}$ like this:

$$a_0=0, \quad a_1=1, \quad a_n=2ka_{n-1}-(k^2+1)a_{n-2} \ (n\geq 2)$$

p is a prime number that $p\equiv 3(\text{mod } 4)$
a) Prove that $a_{n+p^2-1}\equiv a_n(\text{mod } p)$
b) Prove that $a_{n+p^3-p}\equiv a_n(\text{mod } p^2)$

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5 Let $a,b,c\in\mathbb{N}$ be such that $a,b\neq c$. Prove that there are infinitely many prime numbers p for which there exists $n\in\mathbb{N}$ that $p|a^n+b^n-c^n$.

Day 4

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1 We call the set $A\in\mathbb{R}^n$ CN if and only if for every continuous $f:A\to A$ there exists some $x\in A$ such that $f(x)=x$.
a) Example: We know that $A=\{x\in\mathbb{R}^n||x|\leq 1\}$ is CN.
b) The circle is not CN.

Which one of these sets are CN?

1) $A=\{x\in\mathbb{R}^3||x|=1\}$

2) The cross $\{(x,y)\in\mathbb{R}^2|xy=0, \ |x|+|y|\leq 1\}$

3) Graph of the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\sin\frac{1}{x} \text{ if } x\neq 0, \ f(0)=0$$

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2 n vectors are on the plane. We can move each vector forward and backeard on the line that the vector is on it. If there are 2 vectors that their endpoints concide we can omit them and replace them with their sum (If their sum is nonzero). Suppose with these operations with 2 different method we reach to a vector. Prove that these vectors are on a common line

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3 $f(n)$ is the least number that there exist a $f(n)-$mino that contains every $n-$mino.
Prove that $10000\leq f(1384)\leq 960000$.
Find some bound for $f(n)$

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4 a) Year 1872 Texas
3 gold miners found a peice of gold. They have a coin that with possibility of $\frac{1}{2}$ it will come each side, and they want to give the piece of gold to one of themselves depending on how the coin will come. Design a fair method (It means that each of the 3 miners will win the piece of gold with possibility of $\frac{1}{3}$) for the miners.

b) Year 2005, faculty of Mathematics, Sharif university of Technolgy

Suppose $0 < \alpha < 1$ and we want to find a way for people name A and B that the possibity of winning of A is α. Is it possible to find this way?

c) Year 2005 Ahvaz, Takhti Stadium
Two soccer teams have a contest. And we want to choose each player's side with the coin, But we don't know that our coin is fair or not. Find a way to find that coin is fair or not?

d) Year 2005,summer
In the National mathematical Oympiad in Iran. Each student has a coin and must find a way that the possibility of coin being TAIL is α or no. Find a way for the student.

Day 5

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1 An airplane wants to go from a point on the equator, and at each moment it will go to the northeast with speed v. Suppose the radius of earth is R.
a) Will the airplane reach to the north pole? If yes how long it will take to reach the north pole?
b) Will the airplne rotate finitely many times around the north pole? If yes how many times?

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2 We define a relation between subsets of $\mathbb{R}^n$. $A \sim B \iff$ we can partition A, B in sets $A_1, \dots, A_n$ and $B_1, \dots, B_n$(i.e $A = \bigcup_{i=1}^{n} A_i,\ B = \bigcup_{i=1}^{n} B_i, A_i \cap A_j = \emptyset,\ B_i \cap B_j = \emptyset$) and $A_i \simeq B_i$.
Say the the following sets have the relation $\sim$ or not ?

a) Natural numbers and composite numbers.
b) Rational numbers and rational numbers with finite digits in base 10.
c) $\{x \in \mathbb{Q} | x < \sqrt{2}\}$ and $\{x \in \mathbb{Q} | x < \sqrt{3}\}$
d) $A = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ and $A \setminus \{(0, 0)\}$

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3 For each $m \in \mathbb{N}$ we define $rad\ (m) = \prod p_i$, where $m = \prod p_i^{\alpha_i}$.

abc Conjecture
Suppose $\epsilon > 0$ is an arbitrary number, then there exist K depinding on ϵ that for each 3 numbers $a, b, c \in \mathbb{Z}$ that $gcd(a, b) = 1$ and $a + b = c$ then:

$$max\{|a|, |b|, |c|\} \leq K(rad\ (abc))^{1+\epsilon}$$

Now prove each of the following statements by using the abc conjecture :
a) Fermat's last theorem for $n > N$ where N is some natural number.
b) We call $n = \prod p_i^{\alpha_i}$ strong if and only $\alpha_i \geq 2$.
c) Prove that there are finitely many n such that $n,\ n + 1,\ n + 2$ are strong.

d) Prove that there are finitely many rational numbers $\frac{p}{q}$ such that:

$$\left|\sqrt[3]{2}-\frac{p}{q}\right|<\frac{2^{1384}}{q^3}$$

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4 Suppose we have some proteins that each protein is a sequence of 7 "AMINO-ACIDS" $A,\ B,\ C,\ H,\ F,\ N$. For example $AFHNNNHAFFC$ is a protein. There are some steps that in each step an amino-acid will change to another one. For example with the step $NA\to N$ the protein $BANANA$ will cahnge to $BANNA$("in Persian means workman"). We have a set of allowed steps that each protein can change with these steps. For example with the
set of steps:
$1)\ AA\longrightarrow A$
$2)\ AB\longrightarrow BA$
$3)\ A\longrightarrow$ null
Protein $ABBAABA$ will change like this:
$ABB\underline{AA}BA$
$\underline{AB}BABA$
$B\underline{AB}ABA$
$BB\underline{AA}BA$
$BB\underline{AB}A$
$BBB\underline{AA}$
$BBB\underline{A}$
BBB
You see after finite steps this protein will finish it steps.
Set of allowed steps that for them there exist a protein that may have infinitely many steps is dangerous. Which of the following allowed sets are dangerous?
a) $NO\longrightarrow OONN$
b) $\begin{cases} HHCC\longrightarrow HCCH \\ CC\longrightarrow CH \end{cases}$
c) Design a set of allowed steps that change $\underbrace{AA\dots A}_{n}\longrightarrow\underbrace{BB\dots B}_{2^n}$
d) Design a set of allowed steps that change $\underbrace{A\dots A}_{n}\underbrace{B\dots B}_{m}\longrightarrow\underbrace{CC\dots C}_{mn}$
You see from c and d that we acn calculate the functions $F(n)=2^n$ and $G(M,N)=mn$ with these steps. Find some other calculatable functions with these steps. (It has some extra mark.)
