National Math Olympiad (3rd Round) 2006

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–   Number Theory

1 n is a natural number. d is the least natural number that for each a that $gcd(a, n) = 1$ we know $a^d \equiv 1 \pmod{n}$. Prove that there exist a natural number that $\text{ord}_n b = d$

2 n is a natural number that $\frac{x^n+1}{x+1}$ is irreducible over $\mathbb{Z}_2[x]$. Consider a vector in $\mathbb{Z}_2^n$ that it has odd number of 1's (as entries) and at least one of its entries are 0. Prove that these vector and its translations are a basis for $\mathbb{Z}_2^n$

3 L is a fullrank lattice in $\mathbb{R}^2$ and K is a sub-lattice of L, that $\frac{A(K)}{A(L)} = m$. If m is the least number that for each $x \in L$, mx is in K. Prove that there exists a basis $\{x_1, x_2\}$ for L that $\{x_1, mx_2\}$ is a basis for K.

4 a, b, c, t are antural numbers and $k = c^t$ and $n = a^k - b^k$.
a) Prove that if k has at least q different prime divisors, then n has at least qt different prime divisors.
b)Prove that $\varphi(n)$ id divisible by $2^{\frac{t}{2}}$

5   For each n, define $L(n)$ to be the number of natural numbers $1 \le a \le n$ such that $n \mid a^n - 1$. If $p_1, p_2, \ldots, p_k$ are the prime divisors of n, define $T(n)$ as $(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$.
a) Prove that for each $n \in \mathbb{N}$ we have $n \mid L(n)T(n)$.
b) Prove that if $\gcd(n, T(n)) = 1$ then $\varphi(n)|L(n)T(n)$.

6   a) $P(x), R(x)$ are polynomials with rational coefficients and $P(x)$ is not the zero polynomial. Prove that there exist a non-zero polynomial $Q(x) \in \mathbb{Q}[x]$ that

$$P(x) \mid Q(R(x)).$$

b) P, R are polynomial with integer coefficients and P is monic. Prove that there exist a monic polynomial $Q(x) \in \mathbb{Z}[x]$ that
$$P(x) \mid Q(R(x)).$$

–   Algebra

1   For positive numbers $x_1, x_2, \ldots, x_s$, we know that $\prod_{i=1}^{s} x_k = 1$. Prove that for each $m \geq n$

$$\sum_{k=1}^{s} x_k^m \geq \sum_{k=1}^{s} x_k^n$$

2   Find all real polynomials that

$$p(x + p(x)) = p(x) + p(p(x))$$

3   Find all real x, y, z that
$$\begin{cases} x + y + zx = \frac{1}{2} \\ y + z + xy = \frac{1}{2} \\ z + x + yz = \frac{1}{2} \end{cases}$$

4 $p(x)$ is a real polynomial that for each $x \geq 0$, $p(x) \geq 0$. Prove that there are real polynomials $A(x), B(x)$ that $p(x) = A(x)^2 + xB(x)^2$

5   Find the biggest real number k such that for each right-angled triangle with sides a, b, c, we have
$$a^3 + b^3 + c^3 \geq k \left(a + b + c\right)^3.$$

6 P, Q, R are non-zero polynomials that for each $z \in \mathbb{C}$, $P(z)Q(\bar{z}) = R(z)$.
a) If $P, Q, R \in \mathbb{R}[x]$, prove that Q is constant polynomial.
b) Is the above statement correct for $P, Q, R \in \mathbb{C}[x]$?

–   Linear Algebra

1   Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ with $\text{Rank}(A) = k$. Prove that A is sum of k matrices $X_1, \ldots, X_k$ with $\text{Rank}(X_i) = 1$.

2 $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a non-zero linear map. Prove that there is a base $\{v_1, \ldots, v_n m\}$ for $\mathbb{R}^n$ that the set $\{f(v_1), \ldots, f(v_n)\}$ is linearly independent, after ommitting Repetitive elements.

3   Suppose (u, v) is an inner product on $\mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry, that $f(0) = 0$.
1) Prove that for each u, v we have $(u, v) = (f(u), f(v))$
2) Prove that f is linear.

4 $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map, that Image of every $n-1$-dimensional affine space is a $n-1$-dimensional affine space.
1) Prove that Image of every line is a line.
2) Prove that f is an affine map. (i.e. $f = g o h$ that g is a translation and h is a linear map.)

–   Geometry

1   Prove that in triangle ABC, radical center of its excircles lies on line GI, which G is Centroid of triangle ABC, and I is the incenter.

2 ABC is a triangle and R, Q, P are midpoints of AB, AC, BC. Line AP intersects RQ in E and circumcircle of ABC in F. T, S are on RP, PQ such that $ES \perp PQ, ET \perp RP$. F' is on circumcircle of ABC that FF' is diameter. The point of intersection of AF' and BC is E'. S', T' are on AB, AC that $E'S' \perp AB, E'T' \perp AC$. Prove that TS and $T'S'$ are perpendicular.

3   In triangle ABC, if L, M, N are midpoints of AB, AC, BC. And H is orthogonal center of triangle ABC, then prove that

$$LH^2 + MH^2 + NH^2 \leq \frac{1}{4}(AB^2 + AC^2 + BC^2)$$

4   Circle $\Omega(O, R)$ and its chord AB is given. Suppose C is midpoint of arc AB. X is an arbitrary point on the cirlce. Perpendicular from B to CX intersects circle again in D. Perpendicular from C to DX intersects circle again in E. We draw three lines ℓ_1, ℓ_2, ℓ_3 from A, B, E parralell to OX, OD, OC. Prove that these lines are concurrent and find locus of concurrncy point.

5 M is midpoint of side BC of triangle ABC, and I is incenter of triangle ABC, and T is midpoint of arc BC, that does not contain A. Prove that

$$\cos B + \cos C = 1 \Longleftrightarrow MI = MT$$

–   Combinatorics

1   Let A be a family of subsets of $\{1, 2, \ldots, n\}$ such that no member of A is contained in another. Sperners Theorem states that $|A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Find all the families for which the equality holds.

2   Let B be a subset of $\mathbb{Z}_3^n$ with the property that for every two distinct members $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ of B there exist $1 \leq i \leq n$ such that $a_i \equiv b_i + 1 \pmod 3$. Prove that $|B| \leq 2^n$.

3   Let C be a (probably infinite) family of subsets of $\mathbb{N}$ such that for every chain $C_1 \subset C_2 \subset \ldots$ of members of C, there is a member of C containing all of them. Show that there is a member of C such that no other member of C contains it!

4   Let D be a family of s-element subsets of $\{1 \ldots, n\}$ such that every k members of D have non-empty intersection. Denote by $D(n, s, k)$ the maximum cardinality of such a family.
a) Find $D(n, s, 4)$.
b) Find $D(n, s, 3)$.

5   Let E be a family of subsets of $\{1, 2, \ldots, n\}$ with the property that for each $A \subset \{1, 2, \ldots, n\}$ there exist $B \in F$ such that $\frac{n-d}{2} \le |A \triangle B| \le \frac{n+d}{2}$. (where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference). Denote by $f(n, d)$ the minimum cardinality of such a family.
a) Prove that if n is even then $f(n, 0) \le n$.
b) Prove that if $n - d$ is even then $f(n, d) \le \lceil \frac{n}{d+1} \rceil$.
c) Prove that if n is even then $f(n, 0) = n$

6   The National Foundation of Happiness (NFoH) wants to estimate the happiness of people of country. NFoH selected n random persons, and on every morning asked from each of them whether she is happy or not. On any two distinct days, exactly half of the persons gave the same answer. Show that after k days, there were at most $n - \frac{n}{k}$ persons whose yes answers equals their no answers.

–   Final Exam

1   A regular polyhedron is a polyhedron that is convex and all of its faces are regular polygons. We call a regular polhedron a "*Choombam*" iff none of its faces are triangles.
a) prove that each choombam can be inscribed in a sphere.
b) Prove that faces of each choombam are polygons of at most 3 kinds. (i.e. there is a set $\{m, n, q\}$ that each face of a choombam is n-gon or m-gon or q-gon.)
c) Prove that there is only one choombam that its faces are pentagon and hexagon. (Soccer ball)
`http://aycu08.webshots.com/image/5367/2001362702285797426_rs.jpg`
d) For $n > 3$, a prism that its faces are 2 regular n-gons and n squares, is a choombam. Prove that except these choombams there are finitely many choombams.

2   A liquid is moving in an infinite pipe. For each molecule if it is at point with coordinate x then after t seconds it will be at a point of $p(t, x)$. Prove that if $p(t, x)$ is a polynomial of t, x then speed of all molecules are equal and constant.

3   For $A \subset \mathbb{Z}$ and $a, b \in \mathbb{Z}$. We define $aA + b := \{ax + b | x \in A\}$. If $a \ne 0$ then we calll $aA + b$ and A to similar sets. In this question the Cantor set C is the number of non-negative integers that

in their base-3 representation there is no 1 digit. You see

$$C = (3C)\dot{\cup}(3C + 2) \quad (1)$$

(i.e. C is partitioned to sets $3C$ and $3C+2$). We give another example $C = (3C)\dot{\cup}(9C+6)\dot{\cup}(3C+2)$.

A representation of C is a partition of C to some similiar sets. i.e.

$$C = \bigcup_{i=1}^{n} C_i \quad (2)$$

and $C_i = a_i C + b_i$ are similar to C.
We call a representation of C a primitive representation iff union of some of C_i is not a set similar and not equal to C.
Consider a primitive representation of Cantor set. Prove that
a) $a_i > 1$.
b) a_i are powers of 3.
c) $a_i > b_i$
d) (1) is the only primitive representation of C.

4   The image shown below is a cross with length 2. If length of a cross of length k it is called a k-cross. (Each k-cross ahs $6k + 1$ squares.)
`http://aycu08.webshots.com/image/4127/2003057947601864020_th.jpg`
a) Prove that space can be tiled with 1-crosses.
b) Prove that space can be tiled with 2-crosses.
c) Prove that for $k \geq 5$ space can not be tiled with k-crosses.

5   A calculating ruler is a ruler for doing algebric calculations. This ruler has three arms, two of them are sationary and one can move freely right and left. Each of arms is gradient. Gradation of each arm depends on the algebric operation ruler does. For eaxample the ruler below is designed for multiplying two numbers. Gradations are logarithmic.
`http://aycu05.webshots.com/image/5604/2000468517162383885_rs.jpg`
For working with ruler, (e.g for calculating $x.y$) we must move the middle arm that the arrow at the beginning of its gradation locate above the x in the lower arm. We find y in the middle arm, and we will read the number on the upper arm. The number written on the ruler is the answer.
1) Design a ruler for calculating x^y. Grade first arm (x) and (y) from 1 to 10.
2) Find all rulers that do the multiplication in the interval $[1, 10]$.
3) Prove that there is not a ruler for calculating $x^2 + xy + y^2$, that its first and second arm are grade from 0 to 10.

6   Assume that C is a convex subset of $\mathbb{R}^d$. Suppose that $C_1, C_2, \ldots, C_n$ are translations of C that $C_i \cap C \neq \emptyset$ but $C_i \cap C_j = \emptyset$. Prove that

$$n \leq 3^d - 1$$

Prove that $3^d - 1$ is the best bound.

P.S. In the exam problem was given for $n = 3$.

7 We have finite number of distinct shapes in plane. A *"convex Kearting"* of these shapes is covering plane with convex sets, that each set consists exactly one of the shapes, and sets intersect at most in border.

`http://aycu30.webshots.com/image/4109/2003791140004582959_th.jpg`

In which case Convex kearting is possible?

1) Finite distinct points

2) Finite distinct segments

3) Finite distinct circles

8 We mean a traingle in $\mathbb{Q}^n$, 3 points that are not collinear in $\mathbb{Q}^n$

a) Suppose that ABC is triangle in $\mathbb{Q}^n$. Prove that there is a triangle $A'B'C'$ in $\mathbb{Q}^5$ that $\angle B'A'C' = \angle BAC$.

b) Find a natural m that for each traingle that can be embedded in $\mathbb{Q}^n$ it can be embedded in $\mathbb{Q}^m$.

c) Find a triangle that can be embedded in $\mathbb{Q}^n$ and no triangle similar to it can be embedded in $\mathbb{Q}^3$.

d) Find a natural m' that for each traingle that can be embedded in $\mathbb{Q}^n$ then there is a triangle similar to it, that can be embedded in $\mathbb{Q}^m$.

You must prove the problem for $m = 9$ and $m' = 6$ to get complete mark. (Better results leads to additional mark.)