National Math Olympiad (3rd Round) 2007

www.artofproblemsolving.com/community/c3494

by Omid Hatami

– Algebra&Analysis

– August 26th

1 Let a, b be two complex numbers. Prove that roots of $z^4 + az^2 + b$ form a rhombus with origin as center, if and only if $\frac{a^2}{b}$ is a non-positive real number.

2 a, b, c are three different positive real numbers. Prove that:

$$\left|\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a}\right| > 1$$

3 Find the largest real T such that for each non-negative real numbers a, b, c, d, e such that $a+b = c+d+e$:

$$\sqrt{a^2+b^2+c^2+d^2+e^2} \geq T(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}+\sqrt{e})^2$$

4 a) Let $n_1, n_2, \dots$ be a sequence of natural number such that $n_i \geq 2$ and $\epsilon_1, \epsilon_2, \dots$ be a sequence such that $\epsilon_i \in \{1, 2\}$. Prove that the sequence:

$$\sqrt[n_1]{\epsilon_1 + \sqrt[n_2]{\epsilon_2 + \dots + \sqrt[n_k]{\epsilon_k}}}$$

is convergent and its limit is in $(1, 2]$. Define $\sqrt[n_1]{\epsilon_1 + \sqrt[n_2]{\epsilon_2 + \dots}}$ to be this limit.
b) Prove that for each $x \in (1, 2]$ there exist sequences $n_1, n_2, \dots \in \mathbb{N}$ and $n_i \geq 2$ and $\epsilon_1, \epsilon_2, \dots$, such that $n_i \geq 2$ and $\epsilon_i \in \{1, 2\}$, and $x = \sqrt[n_1]{\epsilon_1 + \sqrt[n_2]{\epsilon_2 + \dots}}$

5 Prove that for two non-zero polynomials $f(x, y), g(x, y)$ with real coefficients the system:

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$$

has finitely many solutions in $\mathbb{C}^2$ if and only if $f(x, y)$ and $g(x, y)$ are coprime.

– Geometry

– August 27th

1 Let ABC, l and P be arbitrary triangle, line and point. A', B', C' are reflections of A, B, C in point P. A'' is a point on $B'C'$ such that $AA'' \parallel l$. B'', C'' are defined similarly. Prove that A'', B'', C'' are collinear.

2 a) Let ABC be a triangle, and O be its circumcenter. BO and CO intersect with AC, AB at B', C'. $B'C'$ intersects the circumcircle at two points P, Q. Prove that $AP = AQ$ if and only if ABC is isosceles.
b) Prove the same statement if O is replaced by I, the incenter.

3 Let I be incenter of triangle ABC, M be midpoint of side BC, and T be the intersection point of IM with incircle, in such a way that I is between M and T. Prove that $\angle BIM - \angle CIM = \frac{3}{2}(\angle B - \angle C)$, if and only if $AT \perp BC$.

4 Let ABC be a triangle, and D be a point where incircle touches side BC. M is midpoint of BC, and K is a point on BC such that $AK \perp BC$. Let D' be a point on BC such that $\frac{D'M}{D'K} = \frac{DM}{DK}$. Define ω_a to be circle with diameter DD'. We define ω_B, ω_C similarly. Prove that every two of these circles are tangent.

5 Let ABC be a triangle. Squares AB_cB_aC, CA_bA_cB and BC_aC_bA are outside the triangle. Square $B_cB'_cB'_aB_a$ with center P is outside square AB_cB_aC. Prove that BP, C_aB_a and A_cB_c are concurrent.

– Number Theory

– August 28th

1 Let n be a natural number, such that $(n, 2(2^{1386} - 1)) = 1$. Let $\{a_1, a_2, \dots, a_{\varphi(n)}\}$ be a reduced residue system for n. Prove that:

$$n | a_1^{1386} + a_2^{1386} + \cdots + a_{\varphi(n)}^{1386}$$

2 Let m, n be two integers such that $\varphi(m) = \varphi(n) = c$. Prove that there exist natural numbers $b_1, b_2, \dots, b_c$ such that $\{b_1, b_2, \dots, b_c\}$ is a reduced residue system with both m and n.

3 Let n be a natural number, and $n = 2^{2007}k + 1$, such that k is an odd number. Prove that

$$n \not| 2^{n-1} + 1$$

4 Find all integer solutions of

$$x^4 + y^2 = z^4$$

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5 A hyper-primitive root is a k-tuple $(a_1, a_2, \dots, a_k)$ and $(m_1, m_2, \dots, m_k)$ with the following property:
For each $a \in \mathbb{N}$, that $(a, m) = 1$, has a unique representation in the following form:

$$a \equiv a_1^{\alpha_1} a_2^{\alpha_2} \dots a_k^{\alpha_k} \pmod{m} \qquad 1 \leq \alpha_i \leq m_i$$

Prove that for each m we have a hyper-primitive root.

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6 Something related to this problem (http://www.mathlinks.ro/Forum/viewtopic.php?p=84575\#845756):
Prove that for a set $S \subset \mathbb{N}$, there exists a sequence $\{a_i\}_{i=0}^{\infty}$ in S such that for each n, $\sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$ if and only if $|S| \geq 2$.
By Omid Hatami

– Final Exam

– September 4th

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1 Consider two polygons P and Q. We want to cut P into some smaller polygons and put them together in such a way to obtain Q. We can translate the pieces but we can not rotate them or reflect them. We call P, Q equivalent if and only if we can obtain Q from P(which is obviously an equivalence relation).
http://i3.tinypic.com/4lrb43k.png
a) Let P, Q be two rectangles with the same area(their sides are not necessarily parallel). Prove that P and Q are equivalent.
b) Prove that if two triangles are not translation of each other, they are not equivalent.
c) Find a necessary and sufficient condition for polygons P, Q to be equivalent.

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2 We call the mapping $\Delta : \mathbb{Z} \backslash \{0\} \longrightarrow \mathbb{N}$, a degree mapping if and only if for each $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $b \not| a$ there exist integers r, s such that $a = br + s$, and $\Delta(s) < \Delta(b)$.
a) Prove that the following mapping is a degree mapping:

$$\delta(n) = \text{Number of digits in the binary representation of } n$$

b) Prove that there exist a degree mapping Δ_0 such that for each degree mapping Δ and for each $n \neq 0$, $\Delta_0(n) \leq \Delta(n)$.
c) Prove that $\delta = \Delta_0$
http://i16.tinypic.com/4qntmd0.png

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3 We call a set A a good set if it has the following properties:
1. A consists circles in plane.
2. No two element of A intersect.
Let A, B be two good sets. We say A, B are equivalent if we can reach from A to B by moving circles in A, making them bigger or smaller in such a way that during these operations each circle does not intersect with other circles.
Let a_n be the number of inequivalent good subsets with n elements. For example $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 9$.
`http://i5.tinypic.com/4r0x81v.png`
If there exist a, b such that $Aa^n \leq a_n \leq Bb^n$, we say growth ratio of a_n is larger than a and is smaller than b.
a) Prove that growth ratio of a_n is larger than 2 and is smaller than 4.
b) Find better bounds for upper and lower growth ratio of a_n.

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4 In the following triangular lattice distance of two vertices is length of the shortest path between them. Let $A_1, A_2, \dots, A_n$ be constant vertices of the lattice. We want to find a vertex in the lattice whose sum of distances from vertices is minimum. We start from an arbitrary vertex. At each step we check all six neighbors and if sum of distances from vertices of one of the neighbors is less than sum of distances from vertices at the moment we go to that neighbor. If we have more than one choice we choose arbitrarily, as seen in the attached picture.

Obviusly the algorithm finishes
a) Prove that when we can not make any move we have reached to the problem's answer.
b) Does this algorithm reach to answer for each connected graph?

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5 Look at these fractions. At firs step we have $\frac{0}{1}$ and $\frac{1}{0}$, and at each step we write $\frac{a+b}{c+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$, and we do this forever

$$\begin{array}{ccccccccc}
\frac{0}{1} & & & & & & & & \frac{1}{0} \\
\frac{0}{1} & & & & \frac{1}{1} & & & & \frac{1}{0} \\
\frac{0}{1} & & \frac{1}{2} & & \frac{1}{1} & & \frac{2}{1} & & \frac{1}{0} \\
\frac{0}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1} & \frac{3}{2} & \frac{2}{1} & \frac{3}{1} & \frac{1}{0} \\
 & & & & \dots & & & &
\end{array}$$

a) Prove that each of these fractions is irreducible.
b) In the plane we have put infinitely many circles of diameter 1, over each integer on the real line, one circle. The inductively we put circles that each circle is tangent to two adjacent circles and real line, and we do this forever. Prove that points of tangency of these circles are exactly all the numbers in part a(except $\frac{1}{0}$).
`http://i2.tinypic.com/4m8tmbq.png`
c) Prove that in these two parts all of positive rational numbers appear.
If you don't understand the numbers, look at here (`http://upload.wikimedia.org/wikipedia/commons/2/21/Arabic_numerals-en.svg`).

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6 Scientist have succeeded to find new numbers between real numbers with strong microscopes. Now real numbers are extended in a new larger system we have an order on it (which if induces normal order on $\mathbb{R}$), and also 4 operations addition, multiplication,... and these operation have all properties the same as $\mathbb{R}$.
http://i14.tinypic.com/4tk6mnr.png
a) Prove that in this larger system there is a number which is smaller than each positive integer and is larger than zero.
b) Prove that none of these numbers are root of a polynomial in $\mathbb{R}[x]$.

7 A ring is the area between two circles with the same center, and width of a ring is the difference between the radii of two circles.
http://i18.tinypic.com/6cdmvi8.png
a) Can we put uncountable disjoint rings of width 1(not necessarily same) in the space such that each two of them can not be separated.
http://i19.tinypic.com/4qgx30j.png
b) What's the answer if 1 is replaced with 0?

8 In this question you must make all numbers of a clock, each with using 2, exactly 3 times and Mathematical symbols. You are not allowed to use English alphabets and words like $\sin$ or $\lim$ or a, b and no other digits.
http://i2.tinypic.com/5x73dza.png

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