National Math Olympiad (3rd Round) 2009

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1 Suppose $n > 2$ and let $A_1, \dots, A_n$ be points on the plane such that no three are collinear.
(a) Suppose $M_1, \dots, M_n$ be points on segments $A_1A_2, A_2A_3, \dots, A_nA_1$ respectively. Prove that if $B_1, \dots, B_n$ are points in triangles $M_2A_2M_1, M_3A_3M_2, \dots, M_1A_1M_n$ respectively then

$$|B_1B_2| + |B_2B_3| + \dots + |B_nB_1| \leq |A_1A_2| + |A_2A_3| + \dots + |A_nA_1|$$

Where $|XY|$ means the length of line segment between X and Y.

(b) If X, Y and Z are three points on the plane then by H_{XYZ} we mean the half-plane that it's boundary is the exterior angle bisector of angle $X\hat{Y}Z$ and doesn't contain X and Z ,having Y crossed out.
Prove that if $C_1, \dots, C_n$ are points in $H_{A_nA_1A_2}, H_{A_1A_2A_3}, \dots, H_{A_{n-1}A_nA_1}$ then

$$|A_1A_2| + |A_2A_3| + \dots + |A_nA_1| \leq |C_1C_2| + |C_2C_3| + \dots + |C_nC_1|$$

Time allowed for this problem was 2 hours.

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2 Permutation π of $\{1, \dots, n\}$ is called **stable** if the set $\{\pi(k) - k | k = 1, \dots, n\}$ is consisted of exactly two different elements.
Prove that the number of stable permutation of $\{1, \dots, n\}$ equals to $\sigma(n) - \tau(n)$ in which $\sigma(n)$ is the sum of positive divisors of n and $\tau(n)$ is the number of positive divisors of n.

Time allowed for this problem was 75 minutes.

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3 An arbitary triangle is partitioned to some triangles homothetic with itself. The ratio of homothety of the triangles can be positive or negative.
Prove that sum of all homothety ratios equals to 1.

Time allowed for this problem was 45 minutes.

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4 Does there exists two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that: $\forall x \neq y : |f(x) - f(y)| + |g(x) - g(y)| > 1$

Time allowed for this problem was 75 minutes.

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5 A ball is placed on a plane and a point on the ball is marked.
Our goal is to roll the ball on a polygon in the plane in a way that it comes back to where it started and the marked point comes to the top of it. Note that We are not allowed to rotate without moving, but only rolling.
Prove that it is possible.

Time allowed for this problem was 90 minutes.

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6 Let z be a complex non-zero number such that $Re(z), Im(z) \in \mathbb{Z}$.
Prove that z is uniquely representable as $a_0 + a_1(1+i) + a_2(1+i)^2 + \cdots + a_n(1+i)^n$ where $n \geq 0$ and $a_j \in \{0,1\}$ and $a_n = 1$.

Time allowed for this problem was 1 hour.

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7 A sphere is inscribed in polyhedral P. The faces of P are coloured with black and white in a way that no two black faces share an edge.
Prove that the sum of surface of black faces is less than or equal to the sum of the surface of the white faces.

Time allowed for this problem was 1 hour.

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8 Sone of vertices of the infinite grid $\mathbb{Z}^2$ are missing. Let's take the remainder as a graph. Connect two edges of the graph if they are the same in one component and their other components have a difference equal to one. Call every connected component of this graph a **branch**.
Suppose that for every natural n the number of missing vertices in the $(2n+1) \times (2n+1)$ square centered by the origin is less than $\frac{n}{2}$.
Prove that among the branches of the graph, exactly one has an infinite number of vertices.

Time allowed for this problem was 90 minutes.

– Geometry

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1 1-Let $\triangle ABC$ be a triangle and (O) its circumcircle. D is the midpoint of arc BC which doesn't contain A. We draw a circle W that is tangent internally to (O) at D and tangent to BC.We draw the tangent AT from A to circle $W.P$ is taken on AB such that $AP = AT.P$ and T are at the same side wrt A.PROVE $\angle APD = 90^\circ$.

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2 2-There is given a trapezoid $ABCD$.We have the following properties:$AD||BC, DA = DB = DC, \angle BCD = 72^\circ$. A point K is taken on BD such that $AD = AK, K \neq D$.Let M be the midpoint of $CD.AM$ intersects BD at N.PROVE $BK = ND$.

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3 3-There is given a trapezoid $ABCD$ in the plane with $BC||AD$.We know that the angle bisectors of the angles of the trapezoid are concurrent at O.Let T be the intersection of the diagonals AC, BD.Let Q be on CD such that $\angle OQD = 90^\circ$.Prove that if the circumcircle of the triangle OTQ intersects CD again at P then $TP||AD$.

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4 4-Point P is taken on the segment BC of the scalene triangle ABC such that $AP \neq AB, AP \neq AC.l_1, l_2$ are the incenters of triangles ABP, ACP respectively. circles W_1, W_2 are drawn centered at l_1, l_2 and with radius equal to l_1P, l_2P,respectively. W_1, W_2 intersects at P and Q. W_1 intersects AB and BC at Y_1(the intersection closer to B) and X_1,respectively. W_2 intersects AC

and BC at Y_2(the intersection closer to C$)$ and X_2,respectively.PROVE THE CONCURRENCY OF PQ, X_1Y_1, X_2Y_2.

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5 5-Two circles S_1 and S_2 with equal radius and intersecting at two points are given in the plane.A line l intersects S_1 at B, D and S_2 at A, C(the order of the points on the line are as follows:A, B, C, D).Two circles W_1 and W_2 are drawn such that both of them are tangent externally at S_1 and internally at S_2 and also tangent to l at both sides.Suppose W_1 and W_2 are tangent.Then PROVE $AB = CD$.
