National Math Olympiad (3rd Round) 2010

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Day 1

1 suppose that polynomial $p(x) = x^{2010} \pm x^{2009} \pm ... \pm x \pm 1$ does not have a real root. what is the maximum number of coefficients to be -1?(14 points)

2 a, b, c are positive real numbers. prove the following inequality:

$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{(a+b+c)^2} \geq \frac{7}{25}(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a+b+c})^2$

(20 points)

3 prove that for each natural number n there exist a polynomial with degree $2n+1$ with coefficients in $\mathbb{Q}[x]$ such that it has exactly 2 complex zeros and it's irreducible in $\mathbb{Q}[x]$.(20 points)

4 For each polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ we define it's derivative as this and we show it by $p'(x)$:

$$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + ... + 2 a_2 x + a_1$$

a) For each two polynomials $p(x)$ and $q(x)$ prove that:(3 points)

$$(p(x)q(x))' = p'(x)q(x) + p(x)q'(x)$$

b) Suppose that $p(x)$ is a polynomial with degree n and $x_1, x_2, ..., x_n$ are it's zeros. prove that:(3 points)

$$\frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}$$

c) $p(x)$ is a monic polynomial with degree n and $z_1, z_2, ..., z_n$ are it's zeros such that:

$$|z_1| = 1, \quad \forall i \in \{2, .., n\} : |z_i| \leq 1$$

Prove that $p'(x)$ has at least one zero in the disc with length one with the center z_1 in complex plane. (disc with length one with the center z_1 in complex plane: $D = \{z \in \mathbb{C} : |z - z_1| \leq 1\}$)(20 points)

5 x, y, z are positive real numbers such that $xy+yz+zx=1$. prove that: $3-\sqrt{3}+\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq(x+y+z)^2$
(20 points)

the exam time was 6 hours.

Day 2

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1 suppose that $a=3^{100}$ and $b=5454$. how many zs in $[1,3^{99})$ exist such that for every c that $gcd(c,3)=1$, two equations $x^z\equiv c$ and $x^b\equiv c$ (mod a) have the same number of answers?($\frac{100}{6}$ points)

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2 R is a ring such that $xy=yx$ for every $x,y\in R$ and if $ab=0$ then $a=0$ or $b=0$. if for every Ideal $I\subset R$ there exist $x_1,x_2,..,x_n$ in R (n is not constant) such that $I=(x_1,x_2,...,x_n)$, prove that every element in R that is not 0 and it's not a unit, is the product of finite irreducible elements.($\frac{100}{6}$ points)

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3 If p is a prime number, what is the product of elements like g such that $1\leq g\leq p^2$ and g is a primitive root modulo p but it's not a primitive root modulo p^2, modulo p^2?($\frac{100}{6}$ points)

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4 sppose that $\sigma_k:\mathbb{N}\longrightarrow\mathbb{R}$ is a function such that $\sigma_k(n)=\sum_{d|n}d^k$. $\rho_k:\mathbb{N}\longrightarrow\mathbb{R}$ is a function such that $\rho_k*\sigma_k=\delta$. find a formula for ρ_k.($\frac{100}{6}$ points)

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5 prove that if p is a prime number such that $p=12k+\{2,3,5,7,8,11\}$($k\in\mathbb{N}\cup\{0\}$), there exist a field with p^2 elements.($\frac{100}{6}$ points)

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6 g and n are natural numbers such that $gcd(g^2-g,n)=1$ and $A=\{g^i|i\in\mathbb{N}\}$ and $B=\{x\equiv(n)|x\in A\}$(by $x\equiv(n)$ we mean a number from the set $\{0,1,...,n-1\}$ which is congruent with x modulo n). if for $0\leq i\leq g-1$ $a_i=|[\frac{ni}{g},\frac{n(i+1)}{g})\cap B|$
prove that $g-1|\sum_{i=0}^{g-1}ia_i$.(the symbol | | means the number of elements of the set)($\frac{100}{6}$ points)

the exam time was 4 hours

Day 3

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1 1. In a triangle ABC, O is the circumcenter and I is the incenter. X is the reflection of I to O. A_1 is foot of the perpendicular from X to BC. B_1 and C_1 are defined similarly. prove that AA_1,BB_1 and CC_1 are concurrent.(12 points)

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2 in a quadrilateral $ABCD$, E and F are on BC and AD respectively such that the area of triangles AED and BCF is $\frac{4}{7}$ of the area of $ABCD$. R is the intersection point of digonals of $ABCD$. $\frac{AR}{RC}=\frac{3}{5}$ and $\frac{BR}{RD}=\frac{5}{6}$.

a) in what ratio does EF cut the digonals?(13 points)
b) find $\frac{AF}{FD}$.(5 points)

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3 in a quadrilateral $ABCD$ digonals are perpendicular to each other. let S be the intersection of digonals. K,L,M and N are reflections of S to AB,BC,CD and DA. BN cuts the circumcircle of SKN in E and BM cuts the circumcircle of SLM in F. prove that $EFLK$ is concyclic.(20 points)

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4 in a triangle ABC, I is the incenter. BI and CI cut the circumcircle of ABC at E and F respectively. M is the midpoint of EF. C is a circle with diameter EF. IM cuts C at two points L and K and the arc BC of circumcircle of ABC (not containing A) at D. prove that $\frac{DL}{IL}=\frac{DK}{IK}$.(25 points)

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5 In a triangle ABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. prove that
a) $AP||DM$.(15 points)
b) $AP=2DM$. (10 points)

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6 In a triangle ABC, $\angle C=45$. AD is the altitude of the triangle. X is on AD such that $\angle XBC=90-\angle B$ (X is in the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. if tangent to circumcircle of ABC at M cuts AN at P, prove that P,B and O are collinear.(25 points)

the exam time was 4 hours and 30 minutes.

Day 4

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1 suppose that $\mathcal{F}\subseteq X^{(k)}$ and $|X|=n$. we know that for every three distinct elements of $\mathcal{F}$ like A,B,C, at most one of $A\cap B$,$B\cap C$ and $C\cap A$ is ϕ. for $k\le \frac{n}{2}$ prove that:
a) $|\mathcal{F}|\le max(1,4-\frac{n}{k})\times \binom{n-1}{k-1}$.(15 points)
b) find all cases of equality in a) for $k\le \frac{n}{3}$.(5 points)

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2 suppose that $\mathcal{F}\subseteq \bigcup_{j=k+1}^{n}X^{(j)}$ and $|X|=n$. we know that $\mathcal{F}$ is a sperner family and it's also H_k. prove that: $\sum_{B\in \mathcal{F}}\frac{1}{\binom{n-1}{|B|-1}}\le 1$

(15 points)

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3 suppose that $\mathcal{F}\subseteq p(X)$ and $|X|=n$. we know that for every $A_i,A_j\in \mathcal{F}$ that $A_i\supseteq A_j$ we have $3\le |A_i|-|A_j|$. prove that: $|\mathcal{F}|\le \lfloor \frac{2^n}{3}+\frac{1}{2}\binom{n}{\lfloor \frac{n}{2}\rfloor}\rfloor$

(20 points)

4 suppose that $\mathcal{F}\subseteq X^{(K)}$ and $|X|=n$. we know that for every three distinct elements of $\mathcal{F}$ like A,B and C we have $A\cap B \not\subset C$.

a)(10 points) Prove that :

$$|\mathcal{F}|\le \binom{k}{\lfloor\frac{k}{2}\rfloor}+1$$

b)(15 points) if elements of $\mathcal{F}$ do not necessarily have k elements, with the above conditions show that:

$$|\mathcal{F}|\le \binom{n}{\lceil\frac{n-2}{3}\rceil}+2$$

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5 suppose that $\mathcal{F}\subseteq p(X)$ and $|X|=n$. prove that if $|\mathcal{F}|>\sum_{i=0}^{k-1}\binom{n}{i}$ then there exist $Y\subseteq X$ with $|Y|=k$ such that $p(Y)=\mathcal{F}\cap Y$ that $\mathcal{F}\cap Y=\{F\cap Y:F\in \mathcal{F}\}$(20 points)
you can see this problem also here:
COMBINATORIAL PROBLEMS AND EXERCISES-SECOND EDITION-by LASZLO LOVASZ-AMS CHELSEA PUBLISHING- chapter 13- problem 10(c)!!!

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6 Suppose that X is a set with n elements and $\mathcal{F}\subseteq X^{(k)}$ and $X_1,X_2,...,X_s$ is a partition of X. We know that for every $A,B\in \mathcal{F}$ and every $1\le j\le s$, $E=B\cap(\bigcup_{i=1}^{j}X_i)\neq A\cap(\bigcup_{i=1}^{j}X_i)=F$ shows that none of E,F contains the other one. Prove that

$$|\mathcal{F}|\le \max_{\sum_{i=1}^{S}w_i=k}\prod_{j=1}^{s}\binom{|X_j|}{w_j}$$

(15 points)

Exam time was 5 hours and 20 minutes.

Day 5

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1 **two variable ploynomial**

$P(x,y)$ is a two variable polynomial with real coefficients. degree of a monomial means sum of the powers of x and y in it. we denote by $Q(x,y)$ sum of monomials with the most degree in $P(x,y)$.
(for example if $P(x,y)=3x^4y-2x^2y^3+5xy^2+x-5$ then $Q(x,y)=3x^4y-2x^2y^3$.)
suppose that there are real numbers x_1,y_1,x_2 and y_2 such that $Q(x_1,y_1)>0$, $Q(x_2,y_2)<0$
prove that the set $\{(x,y)|P(x,y)=0\}$ is not bounded.
(we call a set S of plane bounded if there exist positive number M such that the distance of elements of S from the origin is less than M.)

time allowed for this question was 1 hour.

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2 **rolling cube**

a,b and c are natural numbers. we have a $(2a+1)\times(2b+1)\times(2c+1)$ cube. this cube is on an infinite plane with unit squares. you call roll the cube to every side you want. faces of the cube are divided to unit squares and the square in the middle of each face is coloured (it means that if this square goes on a square of the plane, then that square will be coloured.)
prove that if any two of lengths of sides of the cube are relatively prime, then we can colour every square in plane.

time allowed for this question was 1 hour.

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3 **points in plane**

set A containing n points in plane is given. a $copy$ of A is a set of points that is made by using transformation, rotation, homogeneity or their combination on elements of A. we want to put n $copies$ of A in plane, such that every two copies have exactly one point in common and every three of them have no common elements.
a) prove that if no 4 points of A make a parallelogram, you can do this only using transformation. (A doesn't have a parallelogram with angle 0 and a parallelogram that it's two non-adjacent vertices are one!)
b) prove that you can always do this by using a combination of all these things.

time allowed for this question was 1 hour and 30 minutes

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4 **carpeting**

suppose that S is a figure in the plane such that it's border doesn't contain any lattice points. suppose that x,y are two lattice points with the distance 1 (we call a point lattice point if it's coordinates are integers). suppose that we can cover the plane with copies of S such that x,y always go on lattice points (you can rotate or reverse copies of S). prove that the area of S is equal to lattice points inside it.

time allowed for this question was 1 hour.

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5 **interesting sequence**

n is a natural number and $x_1,x_2,...$ is a sequence of numbers 1 and -1 with these properties:

it is periodic and its least period number is 2^n-1. (it means that for every natural number j we have $x_{j+2^n-1}=x_j$ and 2^n-1 is the least number with this property.)

There exist distinct integers $0\le t_1<t_2<...<t_k<n$ such that for every natural number j we have

$$x_{j+n}=x_{j+t_1}\times x_{j+t_2}\times ... \times x_{j+t_k}$$

Prove that for every natural number s that $s < 2^n - 1$ we have

$$\sum_{i=1}^{2^n-1} x_i x_{i+s} = -1$$

Time allowed for this question was 1 hours and 15 minutes.

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6 **polyhedral**

we call a 12-gon in plane good whenever:
first, it should be regular, second, it's inner plane must be filled!!, third, it's center must be the origin of the coordinates, forth, it's vertices must have points $(0,1)$,$(1,0)$,$(-1,0)$ and $(0,-1)$.
find the faces of the massivest polyhedral that it's image on every three plane xy,yz and zx is a good 12-gon.
(it's obvios that centers of these three 12-gons are the origin of coordinates for three dimensions.)

time allowed for this question is 1 hour.

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7 **interesting function**

S is a set with n elements and $P(S)$ is the set of all subsets of S and $f : P(S) \rightarrow \mathbb{N}$
is a function with these properties:
for every subset A of S we have $f(A) = f(S - A)$.
for every two subsets of S like A and B we have $max(f(A), f(B)) \geq f(A \cup B)$
prove that number of natural numbers like x such that there exists $A \subseteq S$ and $f(A) = x$ is less than n.

time allowed for this question was 1 hours and 30 minutes.

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8 [b]numbers $n^2 + 1$[/b]

Prove that there are infinitely many natural numbers of the form $n^2 + 1$ such that they don't have any divisor of the form $k^2 + 1$ except 1 and themselves.

time allowed for this question was 45 minutes.

Day 6

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1 Prove that the group of orientation-preserving symmetries of the cube is isomorphic to S_4 (the group of permutations of $\{1, 2, 3, 4\}$).(20 points)

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2 prove the third sylow theorem: suppose that G is a group and $|G| = p^e m$ which p is a prime number and $(p, m) = 1$. suppose that a is the number of p-sylow subgroups of G ($H < G$ that

$|H| = p^e$). prove that $a|m$ and $p|a-1$.(Hint: you can use this: every two p-sylow subgroups are conjugate.)(20 points)

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3 suppose that $G < S_n$ is a subgroup of permutations of $\{1, ..., n\}$ with this property that for every $e \neq g \in G$ there exist exactly one $k \in \{1, ..., n\}$ such that $g.k = k$. prove that there exist one $k \in \{1, ..., n\}$ such that for every $g \in G$ we have $g.k = k$.(20 points)

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4 a) prove that every discrete subgroup of $(\mathbb{R}^2, +)$ is in one of these forms:
i-$\{0\}$.
ii-$\{mv|m \in \mathbb{Z}\}$ for a vector v in $\mathbb{R}^2$.
iii-$\{mv + nw|m, n \in \mathbb{Z}\}$ for tho linearly independent vectors v and w in $\mathbb{R}^2$.(lattice L)
b) prove that every finite group of symmetries that fixes the origin and the lattice L is in one of these forms: $\mathcal{C}_i$ or $\mathcal{D}_i$ that $i = 1, 2, 3, 4, 6$ ($\mathcal{C}_i$ is the cyclic group of order i and $\mathcal{D}_i$ is the dyhedral group of order i).(20 points)

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5 suppose that p is a prime number. find that smallest n such that there exists a non-abelian group G with $|G| = p^n$.

SL is an acronym for Special Lesson. this year our special lesson was Groups and Symmetries.

the exam time was 5 hours.
