

National Math Olympiad (3rd Round) 2011

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by goodar2006

– Topology

1 (a) We say that a hyperplane H that is given with this equation

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = b\}$$

($a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$ constant) bisects the finite set $A \subseteq \mathbb{R}^n$ if each of the two halfspaces $H^+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n > b\}$ and $H^- = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n < b\}$ have at most $\lfloor \frac{|A|}{2} \rfloor$ points of A .

Suppose that A_1, \dots, A_n are finite subsets of \mathbb{R}^n . Prove that there exists a hyperplane H in \mathbb{R}^n that bisects all of them at the same time.

(b) Suppose that the points in $B = A_1 \cup \dots \cup A_n$ are in general position. Prove that there exists a hyperplane H such that $H^+ \cap A_i$ and $H^- \cap A_i$ contain exactly $\lfloor \frac{|A_i|}{2} \rfloor$ points of A_i .

(c) With the help of part (b), show that the following theorem is true: Two robbers want to divide an open necklace that has d different kinds of stones, where the number of stones of each kind is even, such that each of the robbers receive the same number of stones of each kind. Show that the two robbers can accomplish this by cutting the necklace in at most d places.

2 Prove that these three statements are equivalent:

(a) For every continuous function $f : S^n \rightarrow \mathbb{R}^n$, there exists an $x \in S^n$ such that $f(x) = f(-x)$.

(b) There is no antipodal mapping $f : S^n \rightarrow S^{n-1}$.

(c) For every covering of S^n with closed sets A_0, \dots, A_n , there exists an index i such that $A_i \cap -A_i \neq \emptyset$.

– Combinatorics

1 prove that if graph G is a tree, then there is a vertex that is common between all of the longest paths.

proposed by Sina Rezayi

2 prove that the number of permutations such that the order of each element is a multiple of d is $\frac{n!}{(\frac{n}{d})!d^{\frac{n}{d}}} \prod_{i=0}^{\frac{n}{d}-1} (id + 1)$.

proposed by Mohammad Mansouri

- 3 Suppose that $p(n)$ is the number of partitions of a natural number n . Prove that there exists $c > 0$ such that $P(n) \geq n^{c \cdot \log n}$.

proposed by Mohammad Mansouri

- 4 We say the point i in the permutation σ **ongoing** if for every $j < i$ we have $\sigma(j) < \sigma(i)$.
 a) prove that the number of permutations of the set $\{1, \dots, n\}$ with exactly r ongoing points is $s(n, r)$.

b) prove that the number of n -letter words with letters $\{a_1, \dots, a_k\}$, $a_1 < \dots < a_k$. with exactly r ongoing points is $\sum_m \binom{k}{m} S(n, m) s(m, r)$.

- 5 Suppose that n is a natural number. we call the sequence $(x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2), \dots, (x_s, y_s, z_s, t_s)$ of \mathbb{Z}^4 **good** if it satisfies these three conditions:

- i) $x_1 = y_1 = z_1 = t_1 = 0$.
 ii) the sequences x_i, y_i, z_i, t_i be strictly increasing.
 iii) $x_s + y_s + z_s + t_s = n$. (note that s may vary).

Find the number of good sequences.

proposed by Mohammad Ghiasi

- 6 Every bacterium has a horizontal body with natural length and some nonnegative number of vertical feet, each with nonnegative (!) natural length, that lie below its body. In how many ways can these bacteria fill an $m \times n$ table such that no two of them overlap?

proposed by Mahyar Sefidgaran

– Number Theory

- 1 Suppose that $S \subseteq \mathbb{Z}$ has the following property: if $a, b \in S$, then $a + b \in S$. Further, we know that S has at least one negative element and one positive element. Is the following statement true?

There exists an integer d such that for every $x \in \mathbb{Z}$, $x \in S$ if and only if $d|x$.

proposed by Mahyar Sefidgaran

- 2 Let n and k be two natural numbers such that k is even and for each prime p if $p|n$ then $p - 1|k$. let $\{a_1, \dots, a_{\phi(n)}\}$ be all the numbers coprime to n . What's the remainder of the number $a_1^k + \dots + a_{\phi(n)}^k$ when it's divided by n ?

proposed by Yahya Motevassel

- 3 Let k be a natural number such that $k \geq 7$. How many (x, y) such that $0 \leq x, y < 2^k$ satisfy the equation $73^{73^x} \equiv 9^{9^y} \pmod{2^k}$?

Proposed by Mahyar Sefidgaran

- 4 Suppose that n is a natural number and n is not divisible by 3. Prove that $(n^{2n} + n^n + n + 1)^{2n} + (n^{2n} + n^n + n + 1)^n + 1$ has at least $2d(n)$ distinct prime factors where $d(n)$ is the number of positive divisors of n .

proposed by Mahyar Sefidgaran

- 5 Suppose that k is a natural number. Prove that there exists a prime number in $\mathbb{Z}_{[i]}$ such that every other prime number in $\mathbb{Z}_{[i]}$ has a distance at least k with it.

- 6 a is an integer and p is a prime number and we have $p \geq 17$. Suppose that $S = \{1, 2, \dots, p-1\}$ and $T = \{y \mid 1 \leq y \leq p-1, \text{ord}_p(y) < p-1\}$. Prove that there are at least $4(p-3)(p-1)^{p-4}$ functions $f : S \rightarrow S$ satisfying $\sum_{x \in T} x^{f(x)} \equiv a \pmod{p}$.

proposed by Mahyar Sefidgaran

– Geometry

- 1 We have 4 circles in plane such that any two of them are tangent to each other. we connect the tangency point of two circles to the tangency point of two other circles. Prove that these three lines are concurrent.

proposed by Masoud Nourbakhsh

- 2 In triangle ABC , ω is its circumcircle and O is the center of this circle. Points M and N lie on sides AB and AC respectively. ω and the circumcircle of triangle AMN intersect each other for the second time in Q . Let P be the intersection point of MN and BC . Prove that PQ is tangent to ω iff $OM = ON$.

proposed by Mr.Etesami

- 3 In triangle ABC , X and Y are the tangency points of incircle (with center I) with sides AB and AC respectively. A tangent line to the circumcircle of triangle ABC (with center O) at point A , intersects the extension of BC at D . If D, X and Y are collinear then prove that D, I and O are also collinear.

proposed by Amirhossein Zabeti

- 4 A variant triangle has fixed incircle and circumcircle. Prove that the radical center of its three excircles lies on a fixed circle and the circle's center is the midpoint of the line joining circumcenter and incenter.

proposed by Masoud Nourbakhsh

- 5** Given triangle ABC , D is the foot of the external angle bisector of A , I its incenter and I_a its A -excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A' . Define B' and C' similarly. Prove that AA' , BB' and CC' are concurrent.

proposed by Amirhossein Zabeti

– Algebra

- 1** We define the recursive polynomial $T_n(x)$ as follows: $T_0(x) = 1$, $T_1(x) = x$, $T_{n+1}(x) = 2xT_n(x) + T_{n-1}(x) \forall n \in \mathbb{N}$.

a) find $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$.

b) find all the roots of the polynomial $T_n(x) \forall n \in \mathbb{N}$.

Proposed by Morteza Saghafian

- 2** For nonnegative real numbers x, y, z and t we know that $|x - y| + |y - z| + |z - t| + |t - x| = 4$. Find the minimum of $x^2 + y^2 + z^2 + t^2$.

proposed by Mohammadmahdi Yazdi, Mohammad Ahmadi

- 3** We define the polynomial $f(x)$ in $\mathbb{R}[x]$ as follows: $f(x) = x^n + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \dots + a_1x + a_0$

Prove that there exists an i in the set $\{1, \dots, n\}$ such that we have $|f(i)| \geq \frac{n!}{\binom{n}{i}}$.

proposed by Mohammadmahdi Yazdi

- 4** For positive real numbers a, b and c we have $a + b + c = 3$. Prove $\frac{a}{1+(b+c)^2} + \frac{b}{1+(a+c)^2} + \frac{c}{1+(a+b)^2} \leq \frac{3(a^2+b^2+c^2)}{a^2+b^2+c^2+12abc}$.

proposed by Mohammad Ahmadi

- 5** $f(x)$ is a monic polynomial of degree 2 with integer coefficients such that $f(x)$ doesn't have any real roots and also $f(0)$ is a square-free integer (and is not 1 or -1). Prove that for every integer n the polynomial $f(x^n)$ is irreducible over $\mathbb{Z}[x]$.

proposed by Mohammadmahdi Yazdi

– Final

- 1** A regular dodecahedron is a convex polyhedra that its faces are regular pentagons. The regular dodecahedron has twenty vertices and there are three edges connected to each vertex. Suppose that we have marked ten vertices of the regular dodecahedron.

a) prove that we can rotate the dodecahedron in such a way that at most four marked vertices go to a place that there was a marked vertex before.

b) prove that the number four in previous part can't be replaced with three.

proposed by Kasra Alishahi

2 a) Prove that for every natural numbers n and k , we have monic polynomials of degree n , with integer coefficients like $A = \{P_1(x), \dots, P_k(x)\}$ such that no two of them have a common factor and for every subset of A , the sum of elements of A has all its roots real.

b) Are there infinitely many monic polynomial of degree n with integer coefficients like $P_1(x), P_2(x), \dots$ such that no two of them have a common factor and the sum of a finite number of them has all its roots real?

proposed by Mohammad Mansouri

3 We have connected four metal pieces to each other such that they have formed a tetragon in space and also the angle between two connected metal pieces can vary.

In the case that the tetragon can't be put in the plane, we've marked a point on each of the pieces such that they are all on a plane. Prove that as the tetragon varies, that four points remain on a plane.

proposed by Erfan Salavati

4 The escalator of the station **champion butcher** has this property that if m persons are on it, then its speed is $m^{-\alpha}$ where α is a fixed positive real number.

Suppose that n persons want to go up by the escalator and the width of the stairs is such that all the persons can stand on a stair. If the length of the escalator is l , what's the least time that is needed for these persons to go up? Why?

proposed by Mohammad Ghiasi

5 Suppose that α is a real number and $a_1 < a_2 < \dots$ is a strictly increasing sequence of natural numbers such that for each natural number n we have $a_n \leq n^\alpha$. We call the prime number q golden if there exists a natural number m such that $q|a_m$. Suppose that $q_1 < q_2 < q_3 < \dots$ are all the golden prime numbers of the sequence $\{a_n\}$.

a) Prove that if $\alpha = 1.5$, then $q_n \leq 1390^n$. Can you find a better bound for q_n ?

b) Prove that if $\alpha = 2.4$, then $q_n \leq 1390^{2n}$. Can you find a better bound for q_n ?

[i]part a proposed by mahyar sefidgaran by an idea of this question that the n th prime number is less than 2^{2n-2}

part b proposed by mostafa einollah zade[/i]

- 6 We call two circles in the space fighting if they are intersected or they are clipped. Find a good necessary and sufficient condition for four distinct points A, B, A', B' such that each circle passing through A, B and each circle passing through A', B' are fighting circles.

proposed by Ali Khezeli

- 7 Suppose that $f : P(\mathbb{N}) \rightarrow \mathbb{N}$ and A is a subset of \mathbb{N} . We call f A -predicting if the set $\{x \in \mathbb{N} \mid x \notin A, f(A \cup x) \neq x\}$ is finite. Prove that there exists a function that for every subset A of natural numbers, it's A -predicting.

proposed by Sepehr Ghazi-Nezami

- 8 We call the sequence d_1, \dots, d_n of natural numbers, not necessarily distinct, **covering** if there exists arithmetic progressions like $c_1 + kd_1, \dots, c_n + kd_n$ such that every natural number has come in at least one of them. We call this sequence **short** if we can not delete any of the d_1, \dots, d_n such that the resulting sequence be still covering.

a) Suppose that d_1, \dots, d_n is a short covering sequence and suppose that we've covered all the natural numbers with arithmetic progressions $a_1 + kd_1, \dots, a_n + kd_n$, and suppose that p is a prime number that p divides d_1, \dots, d_k but it does not divide d_{k+1}, \dots, d_n . Prove that the remainders of a_1, \dots, a_k modulo p contains all the numbers $0, 1, \dots, p-1$.

b) Write anything you can about covering sequences and short covering sequences in the case that each of d_1, \dots, d_n has only one prime divisor.

proposed by Ali Khezeli
