National Math Olympiad (3rd Round) 2012

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by goodar2006

– Special Lesson's Exam (First Part)

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1 Prove that the number of incidences of n distinct points on n distinct lines in plane is $\mathcal{O}(n^{\frac{4}{3}})$. Find a configuration for which $\Omega(n^{\frac{4}{3}})$ incidences happens.

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2 Consider a set of n points in plane. Prove that the number of isosceles triangles having their vertices among these n points is $\mathcal{O}(n^{\frac{7}{3}})$. Find a configuration of n points in plane such that the number of equilateral triangles with vertices among these n points is $\Omega(n^2)$.

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3 Prove that if n is large enough, among any n points of plane we can find 1000 points such that these 1000 points have pairwise distinct distances. Can you prove the assertion for n^{α} where α is a positive real number instead of 1000?

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4 Prove that from an $n \times n$ grid, one can find $\Omega(n^{\frac{5}{3}})$ points such that no four of them are vertices of a square with sides parallel to lines of the grid. Imagine yourself as Erdos (!) and guess what is the best exponent instead of $\frac{5}{3}$!

– Special Lesson's Exam (Second Part)

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1 Prove that for each coloring of the points inside or on the boundary of a square with 1391 colors, there exists a monochromatic regular hexagon.

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2 Suppose $W(k,2)$ is the smallest number such that if $n \geq W(k,2)$, for each coloring of the set $\{1,2,...,n\}$ with two colors there exists a monochromatic arithmetic progression of length k. Prove that

$$W(k,2) = \Omega(2^{\frac{k}{2}}).$$

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3 Prove that if n is large enough, then for each coloring of the subsets of the set $\{1,2,...,n\}$ with 1391 colors, two non-empty disjoint subsets A and B exist such that A, B and $A \cup B$ are of the same color.

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4 Prove that if n is large enough, in every $n \times n$ square that a natural number is written on each one of its cells, one can find a subsquare from the main square such that the sum of the numbers is this subsquare is divisible by 1391.

– Number Theory Exam

1 $P(x)$ is a nonzero polynomial with integer coefficients. Prove that there exists infinitely many prime numbers q such that for some natural number n, $q|2^n+P(n)$.

Proposed by Mohammad Gharakhani

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2 Prove that there exists infinitely many pairs of rational numbers $(\frac{p_1}{q},\frac{p_2}{q})$ with $p_1,p_2,q\in\mathbb{N}$ with the following condition:

$$|\sqrt{3}-\frac{p_1}{q}|<q^{-\frac{3}{2}},|\sqrt{2}-\frac{p_2}{q}|<q^{-\frac{3}{2}}.$$

Proposed by Mohammad Gharakhani

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3 p is an odd prime number. Prove that there exists a natural number x such that x and $4x$ are both primitive roots modulo p.

Proposed by Mohammad Gharakhani

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4 $P(x)$ and $Q(x)$ are two polynomials with integer coefficients such that $P(x)|Q(x)^2+1$.

a) Prove that there exists polynomials $A(x)$ and $B(x)$ with rational coefficients and a rational number c such that $P(x)=c(A(x)^2+B(x)^2)$.

b) If $P(x)$ is a monic polynomial with integer coefficients, Prove that there exists two polynomials $A(x)$ and $B(x)$ with integer coefficients such that $P(x)$ can be written in the form of $A(x)^2+B(x)^2$.

Proposed by Mohammad Gharakhani

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5 Let p be a prime number. We know that each natural number can be written in the form

$$\sum_{i=0}^{t}a_ip^i(t,a_i\in\mathbb{N}\cup\{0\},0\le a_i\le p-1)$$

Uniquely.

Now let T be the set of all the sums of the form

$$\sum_{i=0}^{\infty}a_ip^i(0\le a_i\le p-1).$$

(This means to allow numbers with an infinite base p representation). So numbers that for some $N\in\mathbb{N}$ all the coefficients $a_i,i\ge N$ are zero are natural numbers. (In fact we can consider members of T as sequences $(a_0,a_1,a_2,...)$ for which $\forall_{i\in\mathbb{N}}:0\le a_i\le p-1$.) Now we generalize addition and multiplication of natural numbers to this set so that it becomes a ring (it's not necessary to prove this fact). For example:

$1 + (\sum_{i=0}^{\infty}(p-1)p^i) = 1 + (p-1) + (p-1)p + (p-1)p^2 + ... = p + (p-1)p + (p-1)p^2 + ... = p^2 + (p-1)p^2 + (p-1)p^3 + ... = p^3 + (p-1)p^3 + ... = ...$

So in this sum, coefficients of all the numbers $p^k, k \in \mathbb{N}$ are zero, so this sum is zero and thus we can conclude that $\sum_{i=0}^{\infty}(p-1)p^i$ is playing the role of -1 (the additive inverse of 1) in this ring. As an example of multiplication consider

$$(1+p)(1+p+p^2+p^3+...) = 1 + 2p + 2p^2 + \cdots$$

Suppose p is 1 modulo 4. Prove that there exists $x \in T$ such that $x^2 + 1 = 0$.

Proposed by Masoud Shafaei

– Geometry Exam

1 Fixed points B and C are on a fixed circle ω and point A varies on this circle. We call the midpoint of arc BC (not containing A) D and the orthocenter of the triangle ABC, H. Line DH intersects circle ω again in K. Tangent in A to circumcircle of triangle AKH intersects line DH and circle ω again in L and M respectively. Prove that the value of $\frac{AL}{AM}$ is constant.

Proposed by Mehdi E'tesami Fard

2 Let the Nagel point of triangle ABC be N. We draw lines from B and C to N so that these lines intersect sides AC and AB in D and E respectively. M and T are midpoints of segments BE and CD respectively. P is the second intersection point of circumcircles of triangles BEN and CDN. l_1 and l_2 are perpendicular lines to PM and PT in points M and T respectively. Prove that lines l_1 and l_2 intersect on the circumcircle of triangle ABC.

Proposed by Nima Hamidi

3 Cosider ellipse ϵ with two foci A and B such that the lengths of it's major axis and minor axis are $2a$ and $2b$ respectively. From a point T outside of the ellipse, we draw two tangent lines TP and TQ to the ellipse ϵ. Prove that

$$\frac{TP}{TQ} \geq \frac{b}{a}.$$

Proposed by Morteza Saghafian

4 The incircle of triangle ABC for which $AB \neq AC$, is tangent to sides BC, CA and AB in points D, E and F respectively. Perpendicular from D to EF intersects side AB at X, and the second intersection point of circumcircles of triangles AEF and ABC is T. Prove that $TX \perp TF$.

Proposed By Pedram Safaei

5 Two fixed lines l_1 and l_2 are perpendicular to each other at a point Y. Points X and O are on l_2 and both are on one side of line l_1. We draw the circle ω with center O and radius OY. A variable point Z is on line l_1. Line OZ cuts circle ω in P. Parallel to XP from O intersects XZ in S. Find the locus of the point S.

Proposed by Nima Hamidi

– Combinatorics Exam

1 We've colored edges of K_n with $n-1$ colors. We call a vertex rainbow if it's connected to all of the colors. At most how many rainbows can exist?

Proposed by Morteza Saghafian

2 Suppose $s,k,t \in \mathbb{N}$. We've colored each natural number with one of the k colors, such that each color is used infinitely many times. We want to choose a subset $\mathcal{A}$ of $\mathbb{N}$ such that it has t disjoint monochromatic s-element subsets. What is the minimum number of elements of A?

Proposed by Navid Adham

3 In a tree with n vertices, for each vertex x_i, denote the longest paths passing through it by $l_i^1, l_i^2, ..., l_i^{k_i}$. x_i cuts those longest paths into two parts with $(a_i^1, b_i^1), (a_i^2, b_i^2), ..., (a_i^{k_i}, b_i^{k_i})$ vertices respectively. If $\max_{j=1,...,k_i}\{a_i^j \times b_i^j\} = p_i$, find the maximum and minimum values of $\sum_{i=1}^n p_i$.

Proposed by Sina Rezaei

4 **a)** Prove that for all $m,n \in \mathbb{N}$ there exists a natural number a such that if we color every 3-element subset of the set $\mathcal{A} = \{1,2,3,...,a\}$ using 2 colors red and green, there exists an m-element subset of $\mathcal{A}$ such that all 3-element subsets of it are red or there exists an n-element subset of $\mathcal{A}$ such that all 3-element subsets of it are green.

b) Prove that for all $m,n \in \mathbb{N}$ there exists a natural number a such that if we color every k-element subset ($k > 3$) of the set $\mathcal{A} = \{1,2,3,...,a\}$ using 2 colors red and green, there exists an m-element subset of $\mathcal{A}$ such that all k-element subsets of it are red or there exists an n-element subset of $\mathcal{A}$ such that all k-element subsets of it are green.

– Algebra Exam

1 Suppose $0 < m_1 < ... < m_n$ and $m_i \equiv i (\mod 2)$. Prove that the following polynomial has at most n real roots. ($\forall 1 \leq i \leq n : a_i \in \mathbb{R}$).

$$a_0 + a_1 x^{m_1} + a_2 x^{m_2} + ... + a_n x^{m_n}.$$

2 Suppose $N \in \mathbb{N}$ is not a perfect square, hence we know that the continued fraction of $\sqrt{N}$ is of the form $\sqrt{N} = [a_0, \overline{a_1, a_2, ..., a_n}]$. If $a_1 \neq 1$ prove that $a_i \leq 2a_0$.

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3 Suppose p is a prime number and $a, b, c \in \mathbb{Q}^+$ are rational numbers;

a) Prove that $\mathbb{Q}(\sqrt[p]{a} + \sqrt[p]{b}) = \mathbb{Q}(\sqrt[p]{a}, \sqrt[p]{b})$.

b) If $\sqrt[p]{b} \in \mathbb{Q}(\sqrt[p]{a})$, prove that for a nonnegative integer k we have $\sqrt[p]{\frac{b}{a^k}} \in \mathbb{Q}$.

c) If $\sqrt[p]{a} + \sqrt[p]{b} + \sqrt[p]{c} \in \mathbb{Q}$, then prove that numbers $\sqrt[p]{a}$, $\sqrt[p]{b}$ and $\sqrt[p]{c}$ are rational.

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4 Suppose $f(z) = z^n + a_1 z^{n-1} + ... + a_n$ for which $a_1, a_2, ..., a_n \in \mathbb{C}$. Prove that the following polynomial has only one positive real root like α

$$x^n + \Re(a_1)x^{n-1} - |a_2|x^{n-2} - ... - |a_n|$$

and the following polynomial has only one positive real root like β

$$x^n - \Re(a_1)x^{n-1} - |a_2|x^{n-2} - ... - |a_n|.$$

And roots of the polynomial $f(z)$ satisfy $-\beta \leq \Re(z) \leq \alpha$.

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5 Let p be an odd prime number and let $a_1, a_2, ..., a_n \in \mathbb{Q}^+$ be rational numbers. Prove that

$$\mathbb{Q}(\sqrt[p]{a_1} + \sqrt[p]{a_2} + ... + \sqrt[p]{a_n}) = \mathbb{Q}(\sqrt[p]{a_1}, \sqrt[p]{a_2}, ..., \sqrt[p]{a_n}).$$

– Final Exam

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1 Let G be a simple undirected graph with vertices $v_1, v_2, ..., v_n$. We denote the number of acyclic orientations of G with $f(G)$.

a) Prove that $f(G) \leq f(G - v_1) + f(G - v_2) + ... + f(G - v_n)$.

b) Let e be an edge of the graph G. Denote by G' the graph obtained by omiting e and making it's two endpoints as one vertex. Prove that $f(G) = f(G - e) + f(G')$.

c) Prove that for each $\alpha > 1$, there exists a graph G and an edge e of it such that

$\frac{f(G)}{f(G-e)} < \alpha$.

Proposed by Morteza Saghafian

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2 Suppose S is a convex figure in plane with area 10. Consider a chord of length 3 in S and let A and B be two points on this chord which divide it into three equal parts. For a variable point X

in $S-\{A,B\}$, let A' and B' be the intersection points of rays AX and BX with the boundary of S. Let S' be those points X for which $AA' > \frac{1}{3}BB'$. Prove that the area of S' is at least 6.

Proposed by Ali Khezeli

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3 Prove that for each $n \in \mathbb{N}$ there exist natural numbers $a_1 < a_2 < ... < a_n$ such that $\phi(a_1) > \phi(a_2) > ... > \phi(a_n)$.

Proposed by Amirhossein Gorzi

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4 We have n bags each having 100 coins. All of the bags have 10 gram coins except one of them which has 9 gram coins. We have a balance which can show weights of things that have weight of at most 1 kilogram. At least how many times shall we use the balance in order to find the different bag?

Proposed By Hamidreza Ziarati

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5 We call the three variable polynomial P cyclic if $P(x,y,z) = P(y,z,x)$. Prove that cyclic three variable polynomials P_1, P_2, P_3 and P_4 exist such that for each cyclic three variable polynomial P, there exists a four variable polynomial Q such that $P(x,y,z) = Q(P_1(x,y,z), P_2(x,y,z), P_3(x,y,z), P_4(x,$

Solution by Mostafa Eynollahzade and Erfan Salavati

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6 **a)** Prove that $a > 0$ exists such that for each natural number n, there exists a convex n-gon P in plane with lattice points as vertices such that the area of P is less than an^3.

b) Prove that there exists $b > 0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than bn^2.

c) Prove that there exist $\alpha, c > 0$ such that for each natural number n and each n-gon P in plane with lattice points as vertices, the area of P is not less than $cn^{2+\alpha}$.

Proposed by Mostafa Eynollahzade

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7 The city of Bridge Village has some highways. Highways are closed curves that have intersections with each other or themselves in 4-way crossroads. Mr.Bridge Lover, mayor of the city, wants to build a bridge on each crossroad in order to decrease the number of accidents. He wants to build the bridges in such a way that in each highway, cars pass above a bridge and under a bridge alternately. By knowing the number of highways determine that this action is possible or not.

Proposed by Erfan Salavati

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8 **a)** Does there exist an infinite subset S of the natural numbers, such that $S \neq \mathbb{N}$, and such that for each natural number $n \notin S$, exactly n members of S are coprime with n?

b) Does there exist an infinite subset S of the natural numbers, such that for each natural number $n \in S$, exactly n members of S are coprime with n?

Proposed by Morteza Saghafian
