National Math Olympiad (3rd Round) 2013

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by goldeneagle, War-Hammer, TheOverlord

—   Algebra

1   Let $a_0, a_1, \ldots, a_n \in \mathbb{N}$. Prove that there exist positive integers $b_0, b_1, \ldots, b_n$ such that for $0 \le i \le n : a_i \le b_i \le 2a_i$ and polynomial

$$P(x) = b_0 + b_1 x + \cdots + b_n x^n$$

is irreducible over $\mathbb{Q}[x]$.
(10 points)

2   Real numbers $a_1, a_2, \ldots, a_n$ add up to zero. Find the maximum of $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ in term of a_i's, when x_i's vary in real numbers such that $(x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2 \le 1$.
(15 points)

3   For every positive integer $n \ge 2$, Prove that there is no $n-$tuple of distinct complex numbers $(x_1, x_2, \ldots, x_n)$ such that for each $1 \le k \le n$ following equality holds. $\prod_{\substack{1 \le i \le n \\ i \ne k}} (x_k - x_i) = \prod_{\substack{1 \le i \le n \\ i \ne k}} (x_k + x_i)$
(20 points)

4   Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \in \mathbb{Q}$ and

$$f(x + f(y)^2) = f(x + y)^2.$$

(25 points)

5   Prove that there is no polynomial $P \in \mathbb{C}[x]$ such that set $\{P(z) \mid |z| = 1\}$ in complex plane forms a polygon. In other words, a complex polynomial can't map the unit circle to a polygon.
(30 points)

—   Number Theory

1   Let p a prime number and d a divisor of $p - 1$. Find the product of elements in $\mathbb{Z}_p$ with order d. $(\bmod p)$.
(10 points)

2   Suppose that a, b are two odd positive integers such that $2ab + 1 \mid a^2 + b^2 + 1$. Prove that $a = b$.
(15 points)

3   Let $p > 3$ a prime number. Prove that there exist $x, y \in \mathbb{Z}$ such that $p = 2x^2 + 3y^2$ if and only if
$p \equiv 5, 11 \pmod{24}$
(20 points)

4   Prime $p = n^2 + 1$ is given. Find the sets of solutions to the below equation:
$$x^2 - (n^2 + 1)y^2 = n^2.$$

(25 points)

5 $p = 3k+1$ is a prime number. For each $m \in \mathbb{Z}_p$, define function L as follow: $L(m) = \sum_{x \in \mathbb{Z}_p} \left(\frac{x(x^3 + m)}{p} \right)$
a) For every $m \in \mathbb{Z}_p$ and $t \in \mathbb{Z}_p^*$ prove that $L(m) = L(mt^3)$. **(5 points)**

b) Prove that there is a partition of $\mathbb{Z}_p^* = A \cup B \cup C$ such that $|A| = |B| = |C| = \frac{p-1}{3}$ and L on each set is constant.

Equivalently there are a, b, c for which $L(x) = \begin{cases} a & x \in A \\ b & x \in B \\ c & x \in C \end{cases}$. **(7 points)**

c) Prove that $a + b + c = -3$. **(4 points)**

d) Prove that $a^2 + b^2 + c^2 = 6p + 3$. **(12 points)**

e) Let $X = \frac{2a+b+3}{3}, Y = \frac{b-a}{3}$, show that $X, Y \in \mathbb{Z}$ and also show that :$p = X^2 + XY + Y^2$. **(2 points)**

($\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$)

–   Geometry

1   Let $ABCDE$ be a pentagon inscribe in a circle (O). Let $BE \cap AD = T$. Suppose the parallel line with CD which passes through T which cut AB, CE at X, Y. If ω be the circumcircle of triangle AXY then prove that ω is tangent to (O).

2   Let ABC be a triangle with circumcircle (O). Let M, N be the midpoint of arc AB, AC which does not contain C, B and let M', N' be the point of tangency of incircle of $\triangle ABC$ with AB, AC. Suppose that X, Y are foot of perpendicular of A to MM', NN'. If I is the incenter of $\triangle ABC$ then prove that quadrilateral $AXIY$ is cyclic if and only if $b + c = 2a$.

3   Suppose line ℓ and four points A, B, C, D lies on ℓ. Suppose that circles ω_1, ω_2 passes through A, B and circles ω_1', ω_2' passes through C, D. If $\omega_1 \perp \omega_1'$ and $\omega_2 \perp \omega_2'$ then prove that lines O_1O_2', O_2O_1', ℓ are concurrent where O_1, O_2, O_1', O_2' are center of $\omega_1, \omega_2, \omega_1', \omega_2'$.

4   In a triangle ABC with circumcircle (O) suppose that A-altitude cut (O) at D. Let altitude of B, C cut AC, AB at E, F. H is orthocenter and T is midpoint of AH. Parallel line with EF passes through T cut AB, AC at X, Y. Prove that $\angle XDF = \angle YDE$.

5   Let ABC be triangle with circumcircle (O). Let AO cut (O) again at A'. Perpendicular bisector of OA' cut BC at P_A. P_B, P_C define similarly. Prove that :

I) Point P_A, P_B, P_C are collinear.

II) Prove that the distance of O from this line is equal to $\frac{R}{2}$ where R is the radius of the circumcircle.

–   Combinatorics

1   Assume that the following generating function equation is correct, prove the following statement: $\Pi_{i=1}^{\infty}(1 + x^{3i})\Pi_{j=1}^{\infty}(1 - x^{6j+3}) = 1$
Statement: The number of partitions of n to numbers not of the form $6k + 1$ or $6k - 1$ is equal to the number of partitions of n in which each summand appears at least twice.
(10 points)
Proposed by Morteza Saghafian

2   How many rooks can be placed in an $n \times n$ chessboard such that each rook is threatened by at most $2k$ rooks?
(15 points)
Proposed by Mostafa Einollah zadeh

3 n cars are racing. At first they have a particular order. At each moment a car may overtake another car. No two overtaking actions occur at the same time, and except moments a car is passing another, the cars always have an order.
A set of overtaking actions is called "small" if any car overtakes at most once.
A set of overtaking actions is called "complete" if any car overtakes exactly once.
If F is the set of all possible orders of the cars after a small set of overtaking actions and G is the set of all possible orders of the cars after a complete set of overtaking actions, prove that

$$\mid F \mid = 2 \mid G \mid$$

(20 points)
Proposed by Morteza Saghafian

4   We have constructed a rhombus by attaching two equal equilateral triangles. By putting $n - 1$ points on all 3 sides of each triangle we have divided the sides to n equal segments. By drawing line segments between correspounding points on each side of the triangles we have divided the rhombus into $2n^2$ equal triangles.
We write the numbers $1, 2, \ldots, 2n^2$ on these triangles in a way no number appears twice. On the common segment of each two triangles we write the positive difference of the numbers written on those triangles. Find the maximum sum of all numbers written on the segments.

(25 points)
Proposed by Amirali Moinfar

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5   Consider a graph with n vertices and $\frac{7n}{4}$ edges.
(a) Prove that there are two cycles of equal length.
(25 points)
(b) Can you give a smaller function than $\frac{7n}{4}$ that still fits in part (a)? Prove your claim.
We say function $a(n)$ is smaller than $b(n)$ if there exists an N such that for each $n > N$, $a(n) < b(n)$
(At most 5 points)
Proposed by Afrooz Jabal'ameli

–   Final Exam

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1   An n-stick is a connected figure consisting of n matches of length 1 which are placed horizontally or vertically and no two touch each other at points other than their ends. Two shapes that can be transformed into each other by moving, rotating or flipping are considered the same.
An n-mino is a shape which is built by connecting n squares of side length 1 on their sides such that there's a path on the squares between each two squares of the n-mino.
Let S_n be the number of n-sticks and M_n the number of n-minos, e.g. $S_3 = 5$ And $M_3 = 2$.
(a) Prove that for any natural n, $S_n \geq M_{n+1}$.
(b) Prove that for large enough n we have $(2.4)^n \leq S_n \leq (16)^n$.
A **grid segment** is a segment on the plane of length 1 which it's both ends are integer points. A polystick is called **wise** if using it and it's rotations or flips we can cover all grid segments without overlapping, otherwise it's called **unwise**.
(c) Prove that there are at least 2^{n-6} different unwise n-sticks.
(d) Prove that any polystick which is in form of a path only going up and right is wise.
(e) Extra points: Prove that for large enough n we have $3^n \leq S_n \leq 12^n$

Time allowed for this exam was 2 hours.

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2   We define the distance between two circles ω, ω' by the length of the common external tangent of the circles and show it by $d(\omega, \omega')$. If two circles doesn't have a common external tangent then the distance between them is undefined. A point is also a circle with radius 0 and the distance between two cirlces can be zero.
(a) **Centroid.** n circles $\omega_1, \ldots, \omega_n$ are fixed on the plane. Prove that there exists a unique circle $\overline{\omega}$ such that for each circle ω on the plane the square of distance between ω and $\overline{\omega}$ minus the sum of squares of distances of ω from each of the ω_is $1 \leq i \leq n$ is constant, in other words:

$$d(\omega, \overline{\omega})^2 - \frac{1}{n}\sum_{i=1}^{n} d(\omega_i, \omega)^2 = constant$$

(b) **Perpendicular Bisector.** Suppose that the circle ω has the same distance from ω_1, ω_2. Consider ω_3 a circle tangent to both of the common external tangents of ω_1, ω_2. Prove that the

distance of ω from centroid of ω_1, ω_2 is not more than the distance of ω and ω_3. (If the distances are all defined)

(c) **Circumcentre.** Let C be the set of all circles that each of them has the same distance from fixed circles $\omega_1, \omega_2, \omega_3$. Prove that there exists a point on the plane which is the external homothety center of each two elements of C.

(d) **Regular Tetrahedron.** Does there exist 4 circles on the plane which the distance between each two of them equals to 1?

Time allowed for this problem was 150 minutes.

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3   Real function f **generates** real function g if there exists a natural k such that $f^k = g$ and we show this by $f \to g$. In this question we are trying to find some properties for relation $\to$, for example it's trivial that if $f \to g$ and $g \to h$ then $f \to h$.(transitivity)

(a) Give an example of two real functions f, g such that $f \neq g$,$f \to g$ and $g \to f$.
(b) Prove that for each real function f there exists a finite number of real functions g such that $f \to g$ and $g \to f$.
(c) Does there exist a real function g such that no function generates it, except for g itself?
(d) Does there exist a real function which generates both x^3 and x^5?
(e) Prove that if a function generates two polynomials of degree 1 P, Q then there exists a polynomial R of degree 1 which generates P and Q.

Time allowed for this problem was 75 minutes.

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4   A polygon A that doesn't intersect itself and has perimeter p is called **Rotund** if for each two points x, y on the sides of this polygon which their distance on the plane is less than 1 their distance on the polygon is at most $\frac{p}{4}$. (Distance on the polygon is the length of smaller path between two points on the polygon)
Now we shall prove that we can fit a circle with radius $\frac{1}{4}$ in any rotund polygon.
The mathematicians of two planets earth and Tarator have two different approaches to prove the statement. In both approaches by "inner chord" we mean a segment with both endpoints on the polygon, and "diagonal" is an inner chord with vertices of the polygon as the endpoints.
Earth approach: Maximal Chord
We know the fact that for every polygon, there exists an inner chord xy with a length of at most 1 such that for any inner chord $x'y'$ with length of at most 1 the distance on the polygon of x, y is more than the distance on the polygon of x', y'. This chord is called the **maximal chord**.
On the rotund polygon A_0 there's two different situations for maximal chord:
(a) Prove that if the length of the maximal chord is exactly 1, then a semicircle with diameter maximal chord fits completely inside A_0, so we can fit a circle with radius $\frac{1}{4}$ in A_0.
(b) Prove that if the length of the maximal chord is less than one we still can fit a circle with radius $\frac{1}{4}$ in A_0.
Tarator approach: Triangulation
Statement 1: For any polygon that the length of all sides is less than one and no circle with radius $\frac{1}{4}$ fits completely inside it, there exists a triangulation of it using diagonals such that no

diagonal with length more than 1 appears in the triangulation.

Statement 2: For any polygon that no circle with radius $\frac{1}{4}$ fits completely inside it, can be divided into triangles that their sides are inner chords with length of at most 1.

The mathematicians of planet Tarator proved that if the second statement is true, for each rotund polygon there exists a circle with radius $\frac{1}{4}$ that fits completely inside it.

(c) Prove that if the second statement is true, then for each rotund polygon there exists a circle with radius $\frac{1}{4}$ that fits completely inside it.

They found out that if the first statement is true then the second statement is also true, so they put a bounty of a doogh on proving the first statement. A young earth mathematician named J.N., found a counterexample for statement 1, thus receiving the bounty.

(d) Find a 1392-gon that is counterexample for statement 1.

But the Tarators are not disappointed and they are still trying to prove the second statement.

(e) (Extra points) Prove or disprove the second statement.

Time allowed for this problem was 150 minutes.

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5   A subsum of n real numbers $a_1, \ldots, a_n$ is a sum of elements of a subset of the set $\{a_1, \ldots, a_n\}$. In other words a subsum is $\epsilon_1 a_1 + \cdots + \epsilon_n a_n$ in which for each $1 \le i \le n$,ϵ_i is either 0 or 1.

Years ago, there was a valuable list containing n real not necessarily distinct numbers and their $2^n - 1$ subsums. Some mysterious creatures from planet Tarator has stolen the list, but we still have the subsums.

(a) Prove that we can recover the numbers uniquely if all of the subsums are positive.

(b) Prove that we can recover the numbers uniquely if all of the subsums are non-zero.

(c) Prove that there's an example of the subsums for $n = 1392$ such that we can not recover the numbers uniquely.

Note: If a subsum is sum of element of two different subsets, it appears twice.

Time allowed for this question was 75 minutes.

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6   Planet Tarator is a planet in the Yoghurty way galaxy. This planet has a shape of convex 1392-hedron. On earth we don't have any other information about sides of planet tarator.

We have discovered that each side of the planet is a country, and has it's own currency. Each two neighbour countries have their own constant exchange rate, regardless of other exchange rates. Anybody who travels on land and crosses the border must change all his money to the currency of the destination country, and there's no other way to change the money. Incredibly, a person's money may change after crossing some borders and getting back to the point he started, but it's guaranteed that crossing a border and then coming back doesn't change the money.

On a research project a group of tourists were chosen and given same amount of money to travel around the Tarator planet and come back to the point they started. They always travel on land and their path is a nonplanar polygon which doesn't intersect itself. What is the maximum number of tourists that may have a pairwise different final amount of money?

Note 1: Tourists spend no money during travel!

Note 2: The only constant of the problem is 1392, the number of the sides. The exchange rates and the way the sides are arranged are unknown. Answer must be a constant number, regardless of the variables.

Note 3: The maximum must be among all possible polyhedras.

Time allowed for this problem was 90 minutes.

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7 An equation $P(x) = Q(y)$ is called **Interesting** if P and Q are polynomials with degree at least one and integer coefficients and the equations has an infinite number of answers in $\mathbb{N}$.

An interesting equation $P(x) = Q(y)$ **yields in** interesting equation $F(x) = G(y)$ if there exists polynomial $R(x) \in \mathbb{Q}[x]$ such that $F(x) \equiv R(P(x))$ and $G(x) \equiv R(Q(x))$.

(a) Suppose that S is an infinite subset of $\mathbb{N} \times \mathbb{N}$. S *is an answer* of interesting equation $P(x) = Q(y)$ if each element of S is an answer of this equation. Prove that for each S there's an interesting equation $P_0(x) = Q_0(y)$ such that if there exists any interesting equation that S is an answer of it, $P_0(x) = Q_0(y)$ yields in that equation.

(b) Define the degree of an interesting equation $P(x) = Q(y)$ by $max\{deg(P), deg(Q)\}$. An interesting equation is called **primary** if there's no other interesting equation with lower degree that yields in it.

Prove that if $P(x) = Q(y)$ is a primary interesting equation and P and Q are monic then $(deg(P), deg(Q)) = 1$.

Time allowed for this question was 2 hours.

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8 Let $A_1 A_2 A_3 A_4 A_5$ be a convex 5-gon in which the coordinates of all of it's vertices are rational. For each $1 \le i \le 5$ define B_i the intersection of lines $A_{i+1} A_{i+2}$ and $A_{i+3} A_{i+4}$. ($A_i = A_{i+5}$) Prove that at most 3 lines from the lines $A_i B_i$ ($1 \le i \le 5$) are concurrent.

Time allowed for this problem was 75 minutes.