Hungary-Israel Binational 1992

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– Individual

1 Prove that if c is a positive number distinct from 1 and n a positive integer, then

$$n^2 \leq \frac{c^n + c^{-n} - 2}{c + c^{-1} - 2}.$$

2 A set S consists of 1992 positive integers among whose units digits all 10 digits occur. Show that there is such a set S having no nonempty subset S_1 whose sum of elements is divisible by 2000.

3 We are given 100 strictly increasing sequences of positive integers: $A_i = (a_1^{(i)}, a_2^{(i)}, ...), i = 1, 2, ..., 100$. For $1 \leq r, s \leq 100$ we dene the following quantities: $f_r(u) =$ the number of elements of A_r not exceeding n; $f_{r,s}(u) =$ the number of elements of $A_r \cap A_s$ not exceeding n. Suppose that $f_r(n) \geq \frac{1}{2}n$ for all r and n. Prove that there exists a pair of indices (r, s) with $r \neq s$ such that $f_{r,s}(n) \geq \frac{8n}{33}$ for at least ve distinct $n - s$ with $1 \leq n < 19920$.

4 We are given a convex pentagon $ABCDE$ in the coordinate plane such that A, B, C, D, E are lattice points. Let Q denote the convex pentagon bounded by the five diagonals of the pentagon $ABCDE$ (so that the vertices of Q are the interior points of intersection of diagonals of the pentagon $ABCDE$). Prove that there exists a lattice point inside of Q or on the boundary of Q.

– Team

1 We examine the following two sequences: The Fibonacci sequence: $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$; The Lucas sequence: $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. It is known that for all $n \geq 0$

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$. These formulae can be used without proof.

Prove that $1 + L_{2^j} \equiv 0 \pmod{2^{j+1}}$ for $j \geq 0$.

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Prove that

$$\sum_{k=1}^{n} [\alpha^k F_k + \frac{1}{2}] = F_{2n+1} \; \forall n > 1.$$

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We call a nonnegative integer r-Fibonacci number if it is a sum of r (not necessarily distinct) Fibonacci numbers. Show that there innitely many positive integers that are not r-Fibonacci numbers for any $r, 1 \leq r \leq 5$.

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Prove that $F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1} (n \geq 2)$ is not a perfect square.

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$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$. These formulae can be used without proof.

Show that $L_{2n+1} + (-1)^{n+1} (n \geq 1)$ can be written as a product of three (not necessarily distinct) Fibonacci numbers.

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The coordinates of all vertices of a given rectangle are Fibonacci numbers. Suppose that the rectangle is not such that one of its vertices is on the x-axis and another on the y-axis. Prove that either the sides of the rectangle are parallel to the axes, or make an angle of 45° with the axes.
