Hungary-Israel Binational 1993

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–	Individual
–	April 21st
1	Find all pairs of coprime natural numbers a and b such that the fraction $\frac{a}{b}$ is written in the decimal system as $b.a$.
2	Determine all polynomials $f(x)$ with real coeffcients that satisfy $$f(x^2-2x)=f^2(x-2)$$ for all x.
3	Distinct points A,B,C,D,E are given in this order on a semicircle with radius 1. Prove that $$AB^2+BC^2+CD^2+DE^2+AB\cdot BC\cdot CD+BC\cdot CD\cdot DE<4.$$
4	Find the largest possible number of rooks that can be placed on a $3n\times 3n$ chessboard so that each rook is attacked by at most one rook.
–	Team
–	April 22nd
1	In the questions below: G is a nite group; $H\leq G$ a subgroup of G; $\|G:H\|$ the index of H in G; $\|X\|$ the number of elements of $X\subseteq G$; $Z(G)$ the center of G; G' the commutator subgroup of G; $N_G(H)$ the normalizer of H in G; $C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group. Suppose $k\geq 2$ is an integer such that for all $x,y\in G$ and $i\in\{k-1,k,k+1\}$ the relation $(xy)^i=x^iy^i$ holds. Show that G is Abelian.
2	In the questions below: G is a nite group; $H\leq G$ a subgroup of G; $\|G:H\|$ the index of H in G; $\|X\|$ the number of elements of $X\subseteq G$; $Z(G)$ the center of G; G' the commutator subgroup of G; $N_G(H)$ the normalizer of H in G; $C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group.

Suppose that $n \geq 1$ is such that the mapping $x \mapsto x^n$ from G to itself is an isomorphism. Prove that for each $a \in G, a^{n-1} \in Z(G)$.

3 In the questions below: G is a nite group; $H \leq G$ a subgroup of $G; |G:H|$ the index of H in $G; |X|$ the number of elements of $X \subseteq G; Z(G)$ the center of $G; G'$ the commutator subgroup of $G; N_G(H)$ the normalizer of H in $G; C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group.

Show that every element of S_n is a product of 2-cycles.

4 In the questions below: G is a nite group; $H \leq G$ a subgroup of $G; |G:H|$ the index of H in $G; |X|$ the number of elements of $X \subseteq G; Z(G)$ the center of $G; G'$ the commutator subgroup of $G; N_G(H)$ the normalizer of H in $G; C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group.

Let $H \leq G$ and $a, b \in G$. Prove that $|aH \cap Hb|$ is either zero or a divisor of $|H|$.

5 In the questions below: G is a nite group; $H \leq G$ a subgroup of $G; |G:H|$ the index of H in $G; |X|$ the number of elements of $X \subseteq G; Z(G)$ the center of $G; G'$ the commutator subgroup of $G; N_G(H)$ the normalizer of H in $G; C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group.

Let $H \leq G, |H| = 3$. What can be said about $|N_G(H) : C_G(H)|$?

6 In the questions below: G is a nite group; $H \leq G$ a subgroup of $G; |G:H|$ the index of H in $G; |X|$ the number of elements of $X \subseteq G; Z(G)$ the center of $G; G'$ the commutator subgroup of $G; N_G(H)$ the normalizer of H in $G; C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group.

Let $a, b \in G$. Suppose that $ab^2 = b^3a$ and $ba^2 = a^3b$. Prove that $a = b = 1$.

7 In the questions below: G is a nite group; $H \leq G$ a subgroup of $G; |G:H|$ the index of H in $G; |X|$ the number of elements of $X \subseteq G; Z(G)$ the center of $G; G'$ the commutator subgroup of $G; N_G(H)$ the normalizer of H in $G; C_G(H)$ the centralizer of H in G; and S_n the n-th symmetric group.

Assume $|G'| = 2$. Prove that $|G : G'|$ is even.