Hungary-Israel Binational 1998

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Day 1

1 A player is playing the following game. In each turn he ips a coin and guesses the outcome. If his guess is correct, he gains 1 point; otherwise he loses all his points. Initially the player has no points, and plays the game
until he has 2 points.
(a) Find the probability p_n that the game ends after exactly n ips.
(b) What is the expected number of ips needed to nish the game?

2 A triangle ABC is inscribed in a circle with center O and radius R. If the inradii of the triangles OBC, OCA, OAB are r_1, r_2, r_3 , respectively, prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\geq\frac{4\sqrt{3}+6}{R}$.

3 Let a,b,c,m,n be positive integers. Consider the trinomial $f(x)=ax^2+bx+c$. Show that there exist n consecutive natural numbers $a_1,a_2,...,a_n$ such that each of the numbers $f(a_1),f(a_2),...,f(a_n)$ has at least m different prime factors.

Day 2

1 Find all positive integers x and y such that $5^x-3^y=16$.

2 On the sides of a convex hexagon $ABCDEF$, equilateral triangles are constructd in its exterior. Prove that the third vertices of these six triangles are vertices of a regular hexagon if and only if the initial hexagon is *affine regular*. (A hexagon is called affine regular if it is centrally symmetric and any two opposite sides are parallel to the diagonal determine by the remaining two vertices.)

3 Let n be a positive integer. We consider the set P of all partitions of n into a sum of positive integers (the order is irrelevant). For every partition α, let $a_k(\alpha)$ be the number of summands in α that are equal to $k, k=1,2,...,n$. Prove that $\sum_{\alpha\in P}\frac{1}{1^{a_1(\alpha)}a_1(\alpha)!\cdot 2^{a_2(\alpha)}a_2(\alpha)!...n^{a_n(\alpha)}a_n(\alpha)!}=1.$