Hungary-Israel Binational 1999

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Day 1

1 $f(x)$ is a given polynomial whose degree at least 2. Define the following polynomial-sequence: $g_1(x) = f(x), g_{n+1}(x) = f(g_n(x))$, for all $n \in N$. Let r_n be the average of $g_n(x)$'s roots. If $r_{19} = 99$, find r_{99}.

2 $2n + 1$ lines are drawn in the plane, in such a way that every 3 lines define a triangle with no right angles. What is the maximal possible number of acute triangles that can be made in this way?

3 Find all functions $f : \mathbb{Q} \to \mathbb{R}$ that satisfy $f(x + y) = f(x)f(y) - f(xy) + 1$ for every $x, y \in \mathbb{Q}$.

Day 2

1 c is a positive integer. Consider the following recursive sequence: $a_1 = c, a_{n+1} = ca_n + \sqrt{(c^2 - 1)(a_n^2 - 1)}$, for all $n \in N$.
Prove that all the terms of the sequence are positive integers.

2 The function $f(x, y, z) = \frac{x^2+y^2+z^2}{x+y+z}$ is defined for every $x, y, z \in R$ whose sum is not 0. Find a point (x_0, y_0, z_0) such that $0 < x_0^2 + y_0^2 + z_0^2 < \frac{1}{1999}$ and $1.999 < f(x_0, y_0, z_0) < 2$.

3 In a multiple-choice test, there are 4 problems, each having 3 possible answers.
In some group of examinees, it turned out that for every 3 of them, there was a question that each of them gave a different answer to. What is the maximal number of examinees in this group?