Hungary-Israel Binational 2002

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Day 1

1	Find the greatest exponent k for which 2001^k divides $2000^{2001^{2002}} + 2002^{2001^{2000}}$.
2	Points A_1, B_1, C_1 are given inside an equilateral triangle ABC such that $\widehat{B_1AB} = \widehat{A1BA} = 15^0, \widehat{C_1BC} = \widehat{B_1CB} = 20^0, \widehat{A_1CA} = \widehat{C_1AC} = 25^0$. Find the angles of triangle $A_1B_1C_1$.
3	Let $p \geq 5$ be a prime number. Prove that there exists a positive integer $a < p-1$ such that neither of $a^{p-1} - 1$ and $(a+1)^{p-1} - 1$ is divisible by p^2 .

Day 2

1	Suppose that positive numbers x and y satisfy $x^3 + y^4 \leq x^2 + y^3$. Prove that $x^3 + y^3 \leq 2$.
2	Let A', B', C' be the projections of a point M inside a triangle ABC onto the sides BC, CA, AB, respectively. Dene $p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}$. Find the position of point M that maximizes $p(M)$.
3	Let $p(x)$ be a polynomial with rational coefficients, of degree at least 2. Suppose that a sequence (r_n) of rational numbers satises $r_n = p(r_{n+1})$ for every $n \geq 1$. Prove that the sequence (r_n) is periodic.