Hungary-Israel Binational 2003

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Day 1

1	If $x_1, x_2, ..., x_n$ are positive numbers, prove the inequality $\frac{x_1^3}{x_1^2+x_1x_2+x_2^2}+\frac{x_2^3}{x_2^2+x_2x_3+x_3^2}+...+\frac{x_n^3}{x_n^2+x_nx_1+x_1^2}\geq\frac{x_1+x_2+...+x_n}{3}$.
2	Let ABC be an acute-angled triangle. The tangents to its circumcircle at A, B, C form a triangle PQR with $C\in PQ$ and $B\in PR$. Let C_1 be the foot of the altitude from C in ΔABC . Prove that CC_1 bisects $\widehat{QC_1P}$.
3	Let $d>0$ be an arbitrary real number. Consider the set $S_n(d)=\{s=\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}\|x_i\in\mathbb{N}, s<d\}$. Prove that $S_n(d)$ has a maximum element.

Day 2

1	Two players play the following game. They alternately write divisors of $100!$ on the blackboard, not repeating any of the numbers written before. The player after whose move the greatest common divisor of the written numbers equals 1, loses the game. Which player has a winning strategy?
2	Let M be a point inside a triangle ABC . The lines AM, BM, CM intersect BC, CA, AB at A_1, B_1, C_1, respectively. Assume that $S_{MAC_1}+S_{MBA_1}+S_{MCB_1}=S_{MA_1C}+S_{MB_1A}+S_{MC_1B}$. Prove that one of the lines AA_1, BB_1, CC_1 is a median of the triangle ABC.
3	Let n be a positive integer. Show that there exist three distinct integers between n^2 and $n^2+n+3\sqrt{n}$, such that one of them divides the product of the other two.