Hungary-Israel Binational 2009

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by April

Day 1

1 For a given prime $p > 2$ and positive integer k let

$$S_k = 1^k + 2^k + \ldots + (p-1)^k$$

Find those values of k for which $p \,|\, S_k$.

2 Denote the three real roots of the cubic $x^3 - 3x - 1 = 0$ by x_1, x_2, x_3 in order of increasing magnitude. (You may assume that the equation in fact has three distinct real roots.) Prove that $x_3^2 - x_2^2 = x_3 - x_1$.

3 Does there exist a pair $(f;g)$ of strictly monotonic functions, both from $\mathbb{N}$ to $\mathbb{N}$, such that

$$f(g(g(n))) < g(f(n))$$

for every $n \in \mathbb{N}$?

Day 2

1 Given is the convex quadrilateral $ABCD$. Assume that there exists a point P inside the quadrilateral for which the triangles ABP and CDP are both isosceles right triangles with the right angle at the common vertex P. Prove that there exists a point Q for which the triangles BCQ and ADQ are also isosceles right triangles with the right angle at the common vertex Q.

2 Let x, y and z be non negative numbers. Prove that

$$\frac{x^2+y^2+z^2+xy+yz+zx}{6} \le \frac{x+y+z}{3} \cdot \sqrt{\frac{x^2+y^2+z^2}{3}}$$

3 (a) Do there exist 2009 distinct positive integers such that their sum is divisible by each of the given numbers?

(b) Do there exist 2009 distinct positive integers such that their sum is divisible by the sum of any two of the given numbers?

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